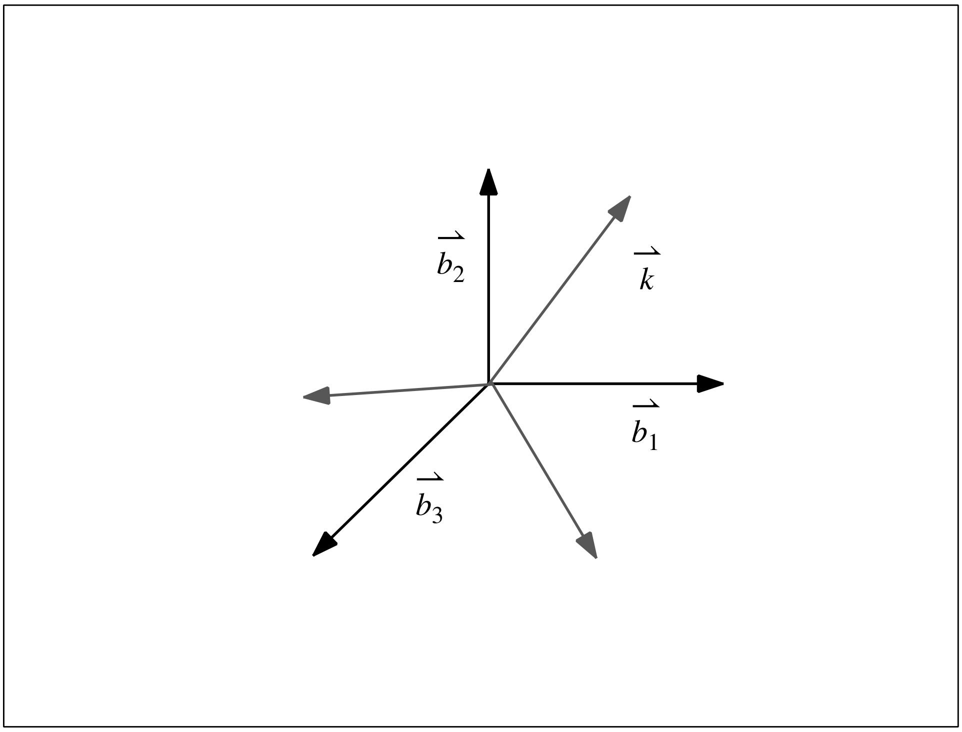
Composing Linear Transformations

Many linear transformations we will want to construct are too complex to construct directly. Most often, we will produce complex linear transformations by composing simpler transformations.

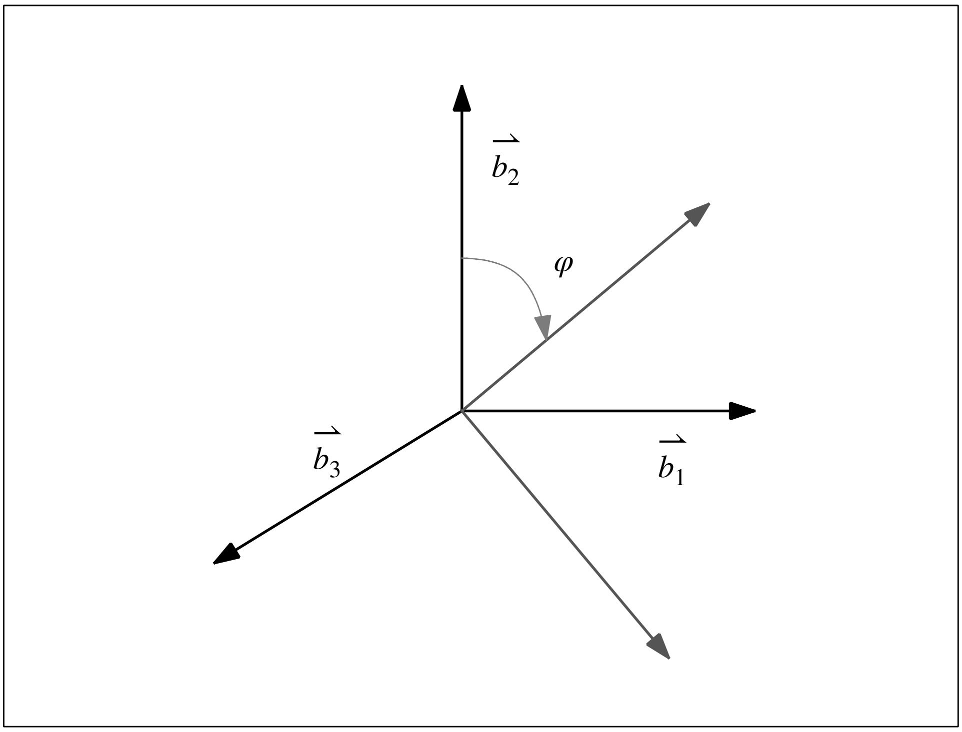
Example: Composing Rotations

The first example I am going to show today is a linear transformation that rotates the coordinate vectors in a standard basis so that the vector $\overrightarrow{b_2}$ ends up aligned with some desired vector $\overrightarrow{k}$ and all the other basis vectors stay aligned perpendicularly.



The most straightforward way to construct this transformation is to think of it as a composition of two rotations.

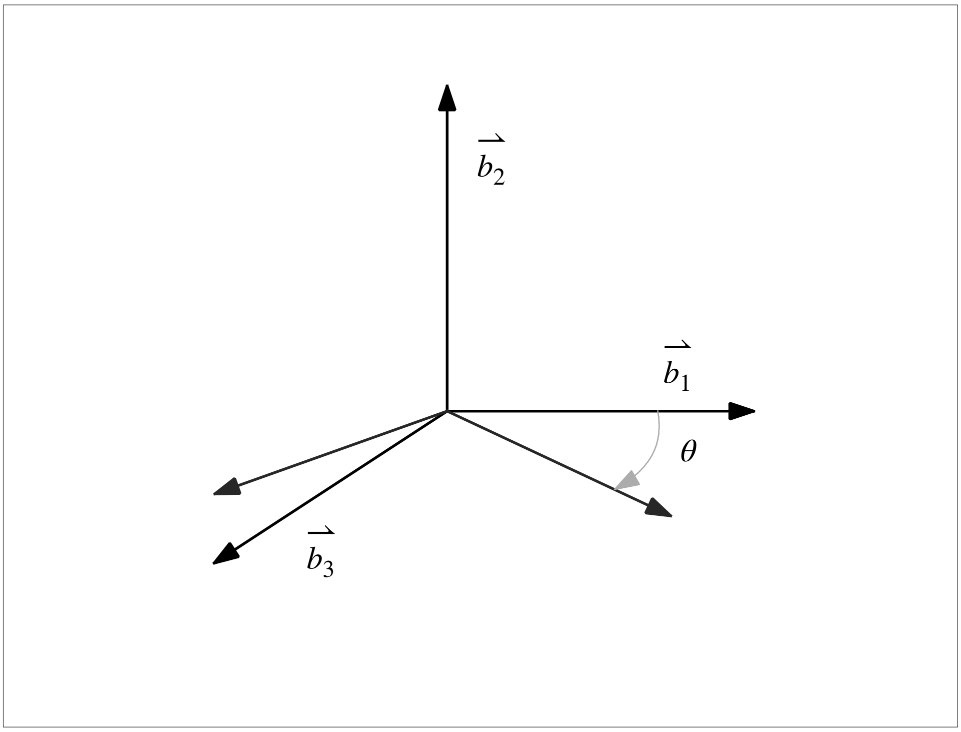
The first rotation is a rotation about the vector $\overrightarrow{b_3}$ that rotates the $\overrightarrow{b_2}$ vector down by an angle φ in the $\overrightarrow{b_1}$, $\overrightarrow{b_2}$ plane.



A little thought will show that this rotation can be represented as

$$L_\varphi(\vec{v}) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Next, consider a rotation about the $\vec{b}_2$ vector through an angle θ that looks like this.



A little thought will show that this rotation can be represented as

$$L_\theta(\vec{v}) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

How do we compose these two rotations? The correct approach here is to think of the first transformation in this way:

$$L_\varphi(\vec{v}) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \left(\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right)$$

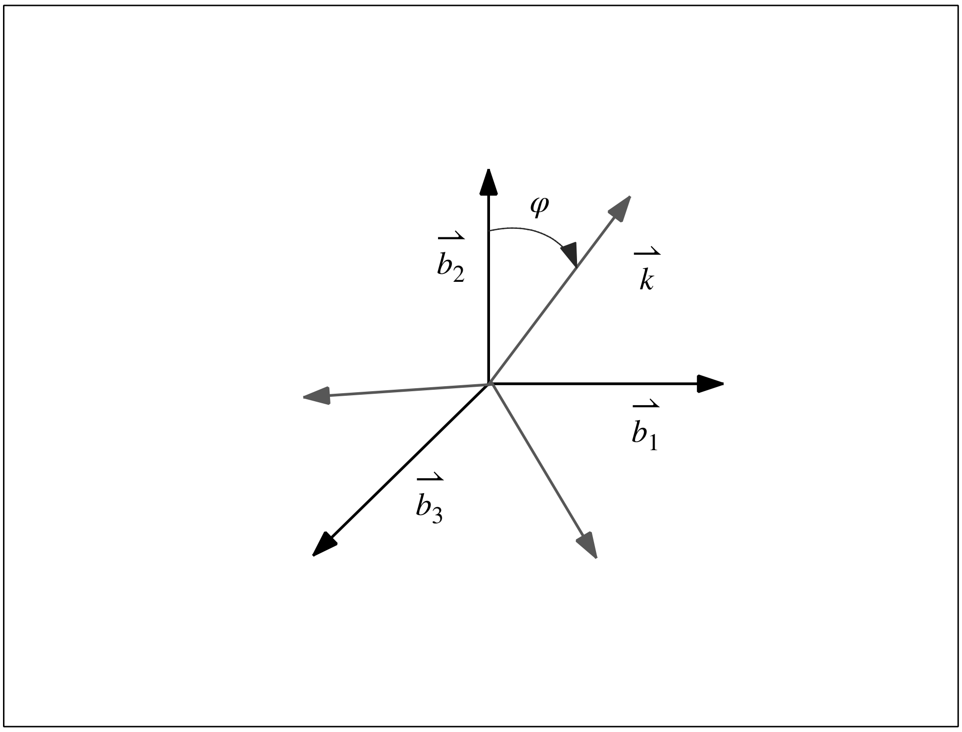
Viewed this way, the first transformation moves vectors to a new set of coordinates, while leaving the basis unchanged. It is essential that we leave the basis unchanged, because the second transformation is going to be performed *relative to that same basis*.

$$L_\theta\left(L_\varphi(\vec{v})\right) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \left(\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \cos(\varphi)\cos(\theta) & \sin(\varphi)\cos(\theta) & -\sin(\theta) \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ \cos(\varphi)\sin(\theta) & \sin(\varphi)\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

The final question here is how to select the angles φ and θ so that the vector $\vec{b}_2$ ends up aligned with some desired vector $\vec{k}$.

The key observation is that after the second rotation the angle between $\vec{k}$ and $\vec{b}_2$ is still φ:

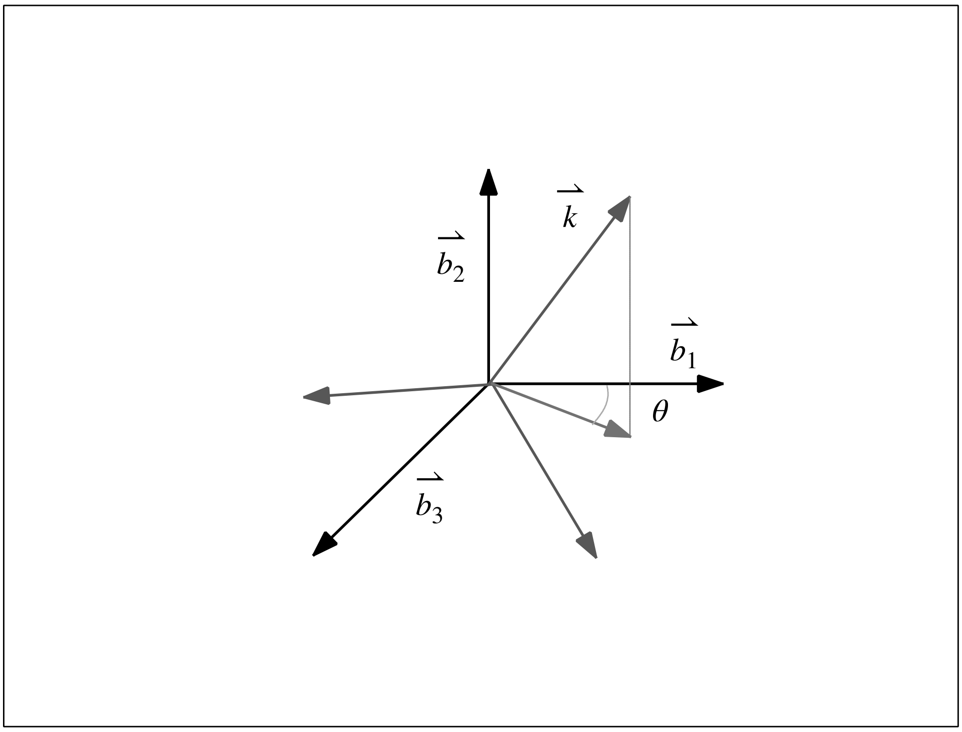


We can compute φ by making use of a key fact about the dot product:

$$\vec{k} \cdot \vec{b}_2 = |\vec{k}|\,|\vec{b}_2|\cos\varphi$$

$$\varphi = \cos^{-1}\left(\frac{\vec{k} \cdot \vec{b}_2}{|\vec{k}|\,|\vec{b}_2|}\right) = \cos^{-1}\left(\frac{k_2}{|\vec{k}|}\right)$$

We can compute θ by noting that if we project $\vec{k}$ down to the $\vec{b}_1$, $\vec{b}_3$ plane the projection will make an angle θ with $\vec{b}_1$.



The computation of the angle θ is left as an exercise for the reader.

Order of transformations

An important observation about multiple transformations is that the order of the transformations matters. To see this, imagine taking the transformation above

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

and switching the order of the two transformations:

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Can you come up with a simple example to show that this produces a different result?

Another, more interesting, question comes from taking grouping into account. The original transformation can be grouped this way:

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \left(\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \left(\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) \right)$$

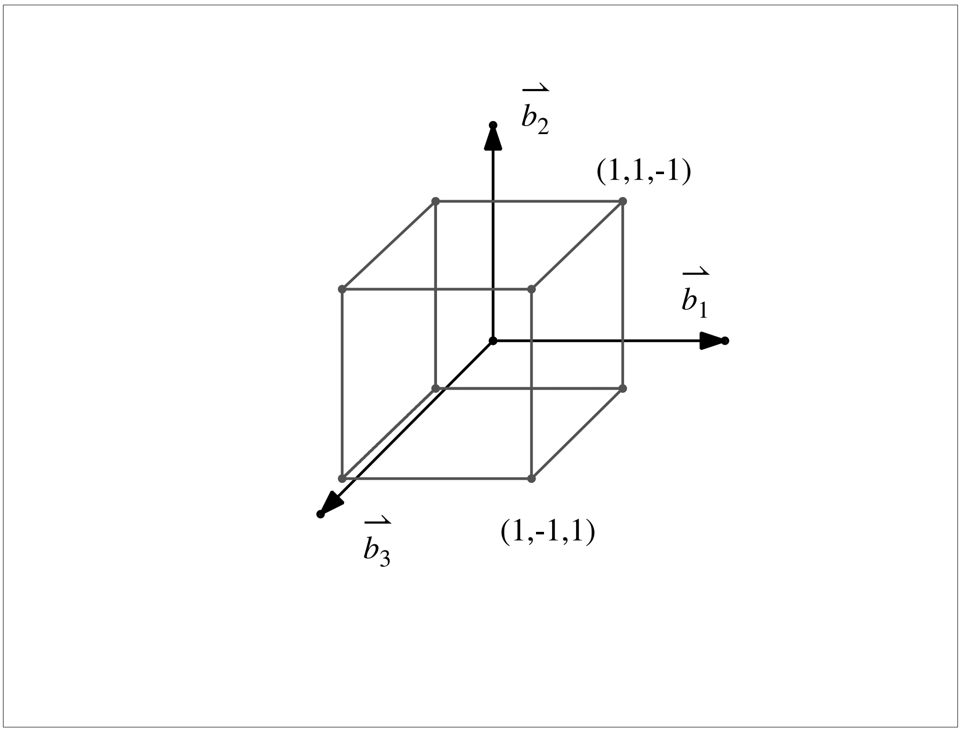
The alternative transformation can be grouped this way:

$$\left(\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix}\begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\right)\left(\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix}\right)$$

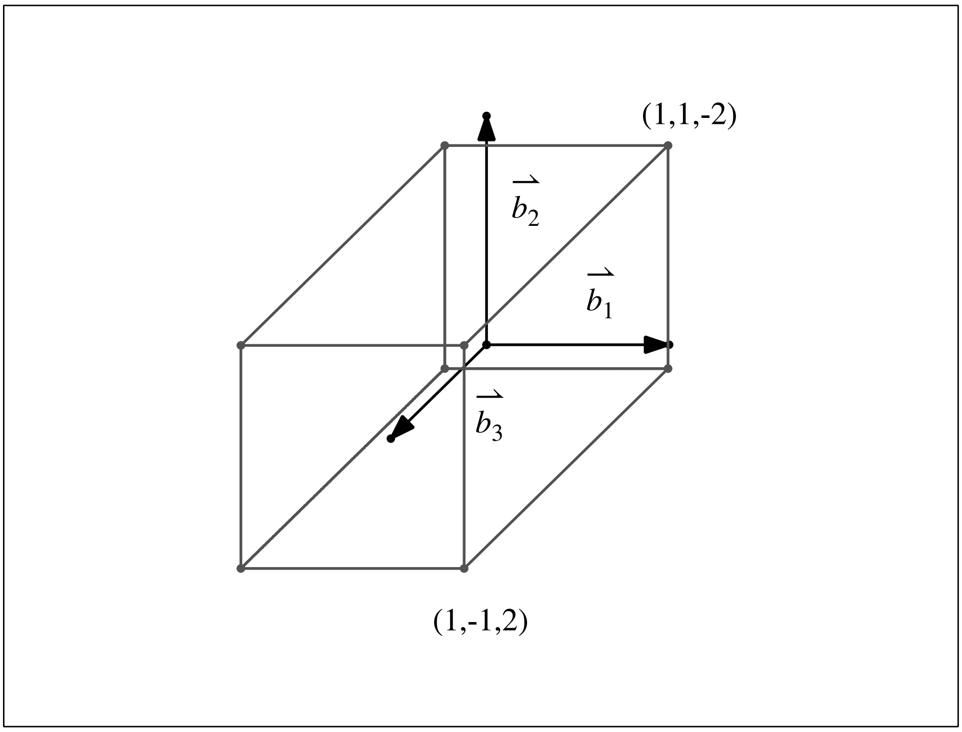
Does this grouping provide you with an alternative perspective on what is going on? If not, try re-reading chapter 2.

Second Example

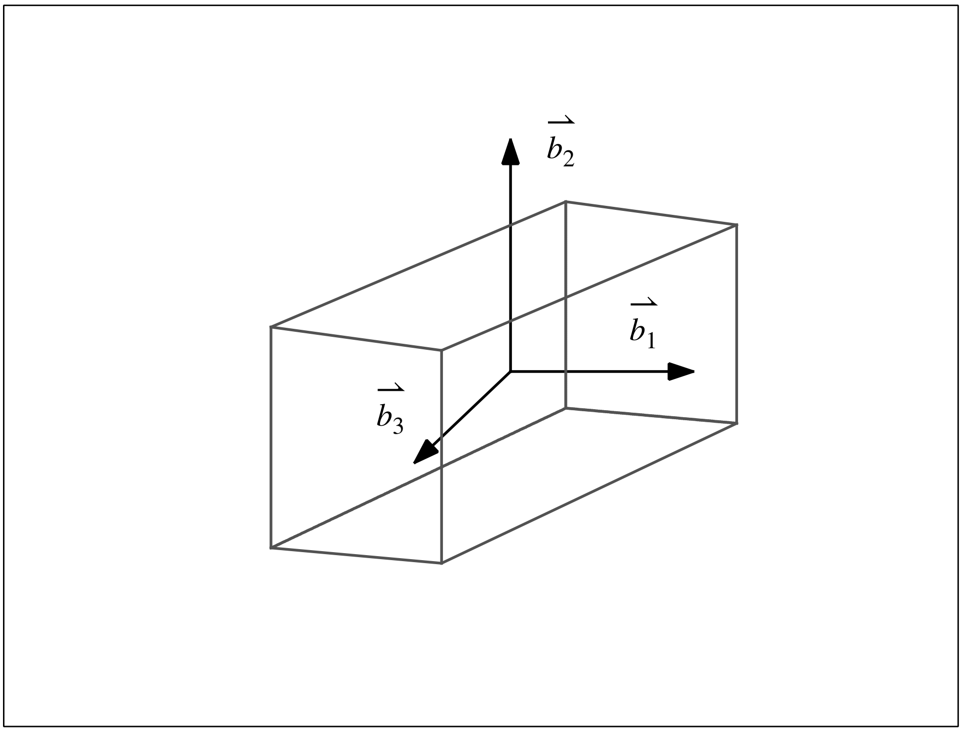
For our second example we are going to compose a scaling transformation with a rotation. Specifically, I want to start with a standard cube



stretch the cube to turn it into something that looks more like this



and finally rotate the stretched shape about the $\vec{b_2}$ axis.



The initial stretching transformation is accomplished by scaling coordinates in the $\vec{b_3}$ direction up by a factor of two:

$$L_2(\vec{v}) = \left[\vec{b_1}\ \vec{b_2}\ \vec{b_3}\right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

The rotation is represented as

$$L_\theta(\vec{v}) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

These two transformations compose as

$$L_\theta\left(L_2(\vec{v})\right) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & -2\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & 2\cos(\theta) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$