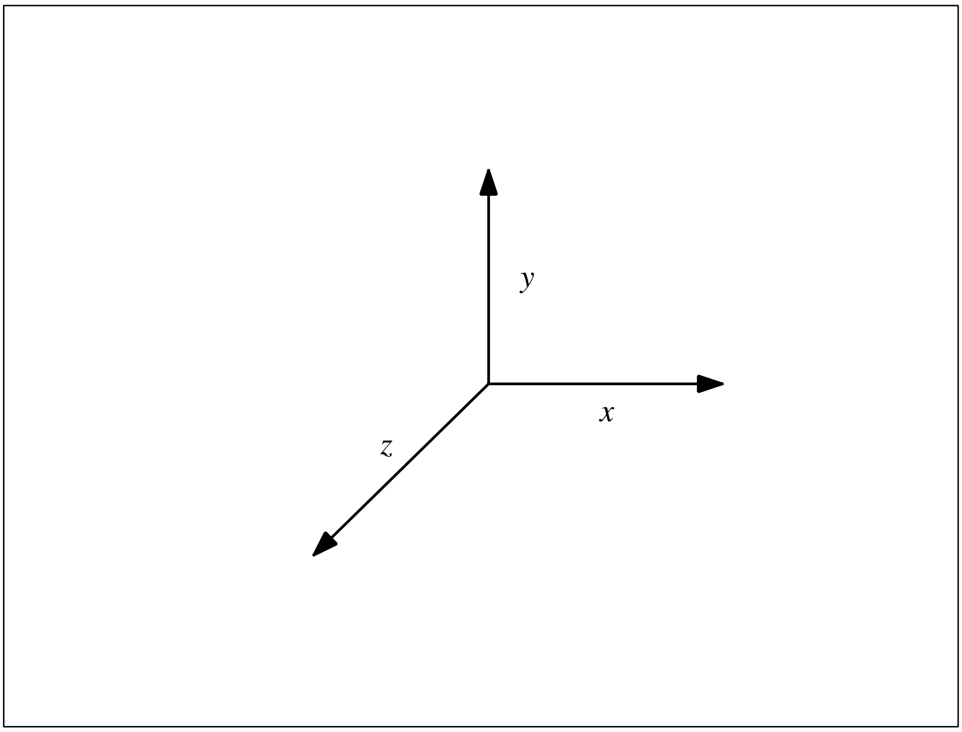
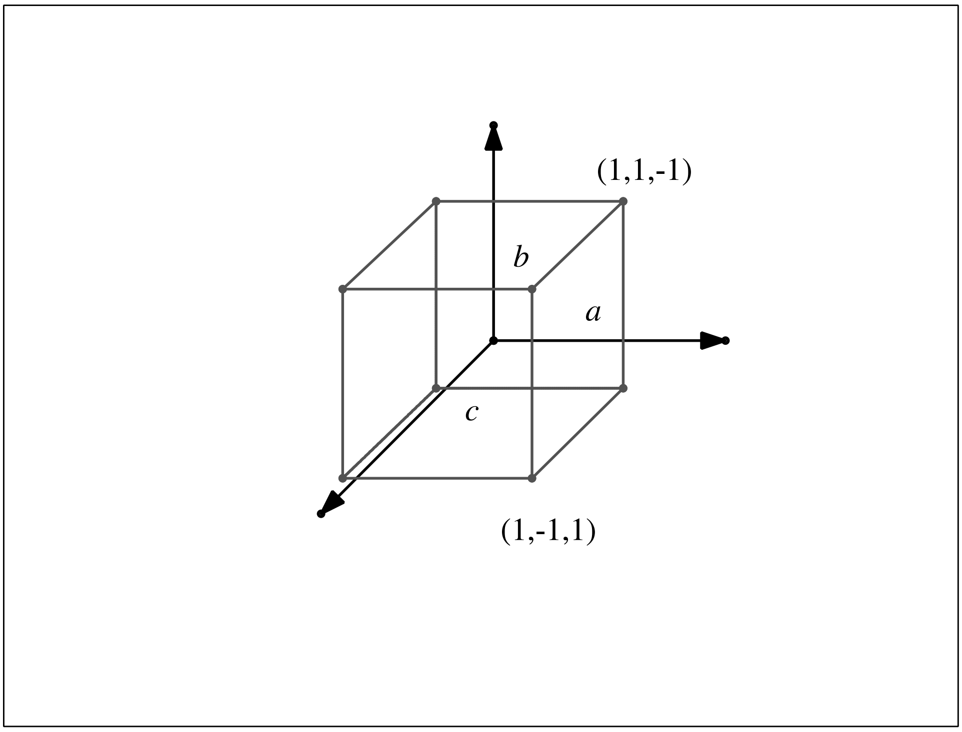
The world coordinate system

All drawing in 3-d OpenGL takes place within the context of a world coordinate system.

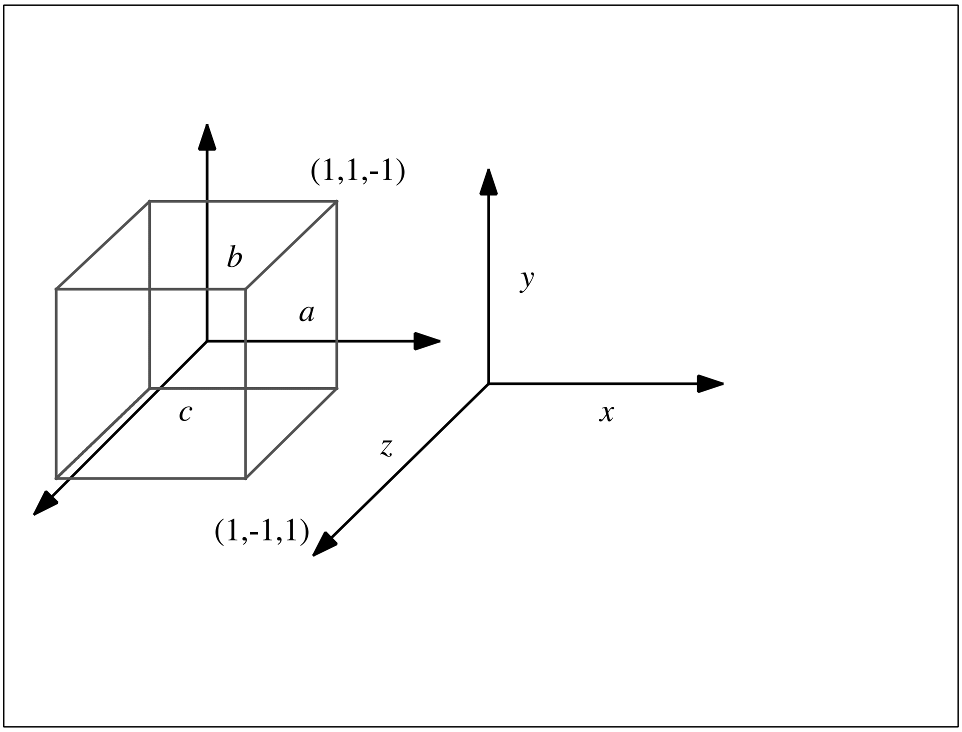


Object coordinate systems

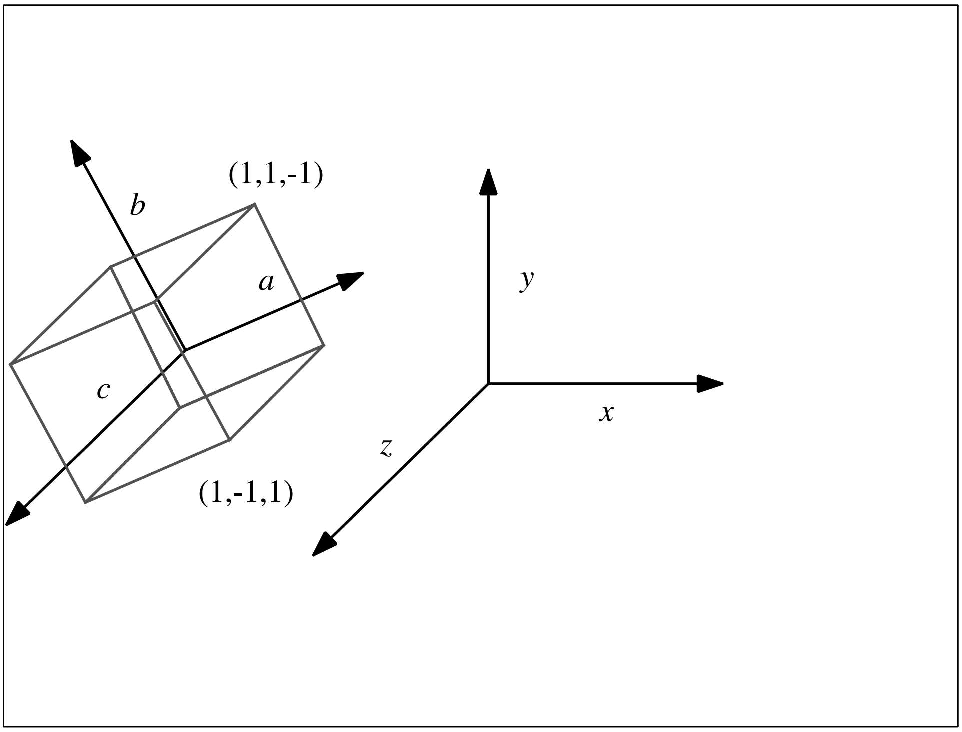
When objects are created in our 3-d world, they start as a simple drawing relative to some convenient *object coordinate system.*



By adjusting the position of the object coordinate system's origin relative to the world coordinate system we can place the object in the world.



By changing the orientation of the object coordinate system's axes relative to the world coordinate system's axes we can change the apparent orientation of the object.



Notation

The basic mathematical objects we will be working with are points and vectors. I will use the notation

$$\tilde{p}$$

for points and

$$\vec{v}$$

for vectors. In addition, we will frequently need to work with lists of things. We will use the notational convention of using boldface type for variables that represent a list. For example, a variable that represents a vertical list of vectors looks like

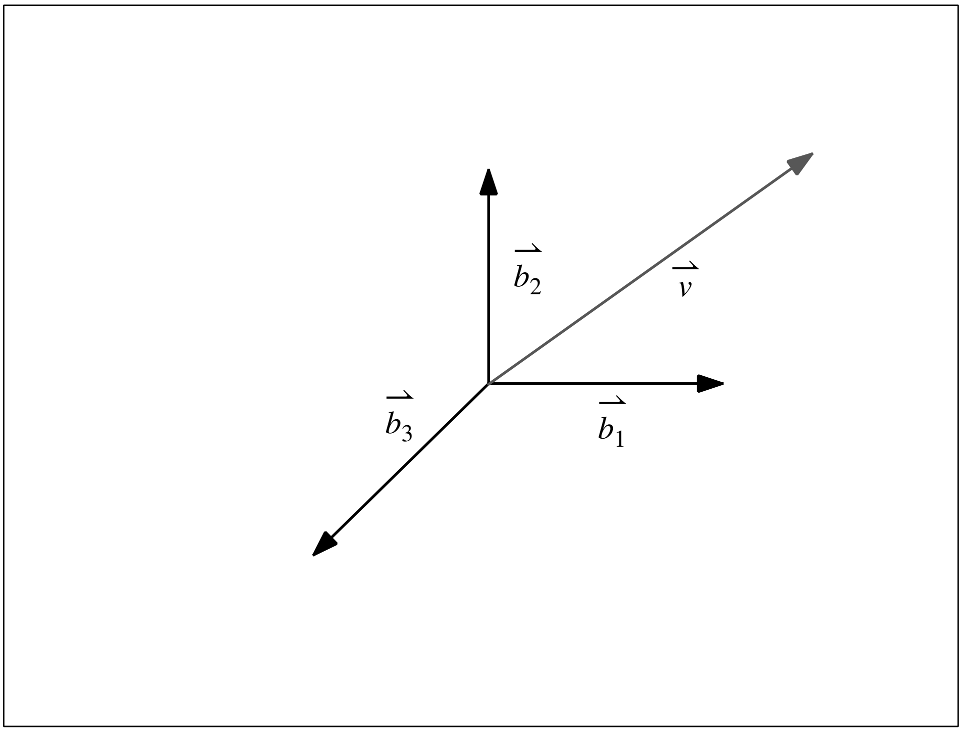
$$\vec{\mathbf{b}}$$

A horizontal list is just a vertical list transposed. For this we use the notation

$$\vec{\mathbf{b}}^t$$

Coordinate representations

Frequently in this course we will have a need to represent one thing as a combination of other things. The first time we encounter this concept is in the context of *coordinate representations of vectors*. The basic idea here is to write an arbitrary vector as a combination of a set of simpler, "standard" vectors.



We write the vector $\vec{v}$ as a linear combination of the *coordinate vectors* $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$:

$$\vec{v} = c_1 \vec{b_1} + c_2 \vec{b_2} + c_3 \vec{b_3}$$

Using list notation we can also write this as

$$\vec{v} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

We can further condense the notation here by using boldface to indicate that some of these things are lists.

$$\vec{v} = \vec{\mathbf{b}}^t \ \mathbf{c}$$

The notation here indicates that $\mathbf{c}$ is a vertical list of numbers, while $\vec{\mathbf{b}}^t$ is a horizontal list of vectors.

If the coordinate vectors $\vec{b_1}$, $\vec{b_2}$, and $\vec{b_3}$ are mutually perpendicular and all have length 1, we say that they form an *orthonormal basis* of vectors. The list of numbers

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

is called a list of *coordinates*. We then refer to the product $\vec{\mathbf{b}}^t \ \mathbf{c}$ as the *coordinate representation* of the vector $\vec{v}$ with respect to the basis $\vec{\mathbf{b}}^t$.

Transformations

A *linear transformation* is a mapping from one vector space to another (or of a vector space to itself) that satisfies a *linearity condition*. If $\vec{v_1}$ and $\vec{v_2}$ are any two vectors and c_1 and c_2 are any two constants, the mapping L is linear if

$$L\left(c_1 \vec{v_1} + c_2 \vec{v_2}\right) = c_1 L\left(\vec{v_1}\right) + c_2 L\left(\vec{v_2}\right)$$

Since we are already in the habit of representing vectors as combinations of a set of basis vectors, we will extend that approach to linear transformations. Given a linear transformation L we will seek a *representation of L with respect to the basis*.

Here is how this will work. To start with, consider any arbitrary vector, $\vec{v}$. As before, we seek to represent that vector with respect to some basis.

$$\vec{v} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Now consider what happens when we hit $\vec{v}$ with the linear transformation:

$$L(\vec{v}) = L\left(\begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\right) = L\left(c_1 \vec{b_1} + c_2 \vec{b_2} + c_3 \vec{b_3}\right)$$

$$= c_1 L\left(\vec{b_1}\right) + c_2 \; L\left(\vec{b_2}\right) + c_3 L\left(\vec{b_3}\right)$$

$$= \begin{bmatrix} L\left(\vec{b_1}\right) & L\left(\vec{b_2}\right) & L\left(\vec{b_3}\right) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

This string of equalities illustrates an important principle.

To understand the action of a linear transformation, try to understand what the transformation does to the basis vectors.

Representing a linear transformation with respect to a basis

Let us now push the last set of equations a little further.

$$L(\vec{v}) = \begin{bmatrix} L\left(\vec{b_1}\right) & L\left(\vec{b_2}\right) & L\left(\vec{b_3}\right) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

I observed earlier that any vector can be written as a combination of basis vectors. Let us now apply that idea on the right hand side.

$$L\left(\vec{b_1}\right) = M_{1,1} \vec{b_1} + M_{2,1} \vec{b_2} + M_{3,1} \vec{b_3} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix}\begin{bmatrix} M_{1,1} \\ M_{2,1} \\ M_{3,1} \end{bmatrix}$$

$$L\left(\vec{b_2}\right) = M_{1,2} \vec{b_1} + M_{2,2} \vec{b_2} + M_{3,2} \vec{b_3} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix}\begin{bmatrix} M_{1,2} \\ M_{2,2} \\ M_{3,2} \end{bmatrix}$$

$$L\left(\vec{b_1}\right) = M_{1,3} \vec{b_1} + M_{2,3} \vec{b_2} + M_{3,3} \vec{b_3} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix}\begin{bmatrix} M_{1,3} \\ M_{2,3} \\ M_{3,3} \end{bmatrix}$$

Putting all of this together with the equation above gives

$$L(\vec{v}) = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix}\begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

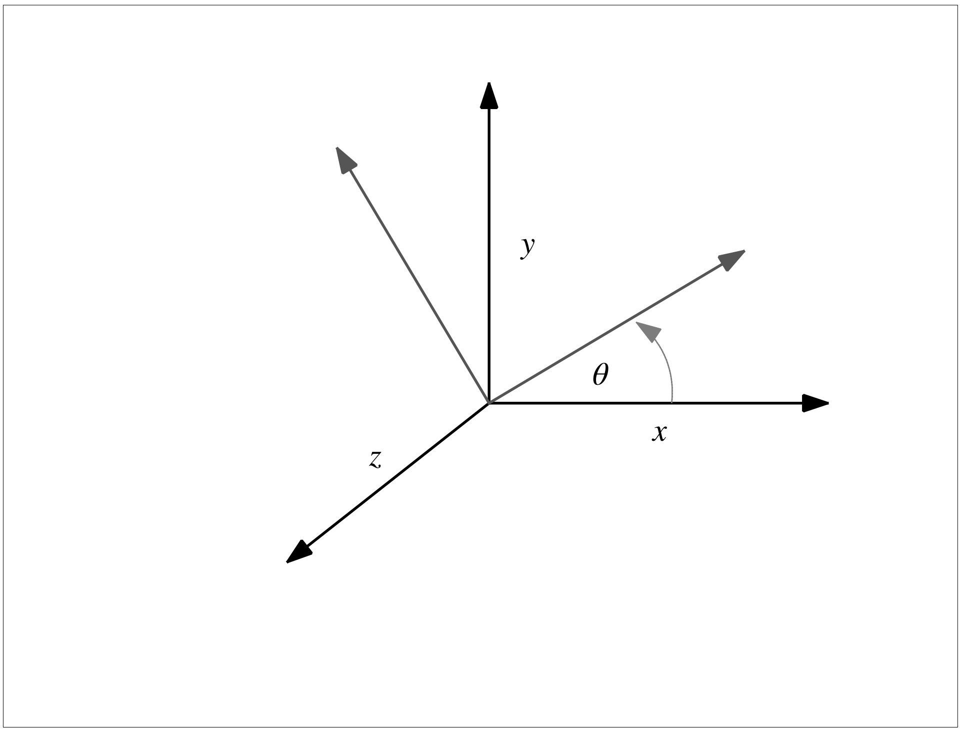
The matrix

$$M = \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix}$$

is the *representation of the linear transformation L with respect to the basis* $\overrightarrow{\mathbf{b}}^t$.

An example

Let us now apply the procedure outlined above to construct the matrix representation for an example linear transformation. Let us construct the matrix representation for a transformation R_θ that rotates vectors through a counter-clockwise angle of θ radians about the z-axis.



For this example we will use the standard basis:

$$\overrightarrow{b_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\overrightarrow{b_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{b_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We compute

$$L\left(\vec{b_1}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

$$L\left(\vec{b_2}\right) = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

$$L\left(\vec{b_3}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Putting this together we have

$$L(\vec{v}) = \begin{bmatrix} L\left(\vec{b_1}\right) & L\left(\vec{b_2}\right) & L\left(\vec{b_3}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Similar reasoning leads to matrices for rotations about the *x* and *y* axes.