Constructing the basis functions

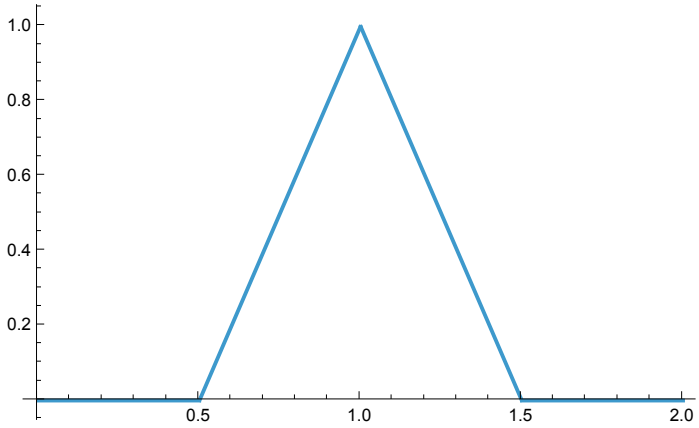
The basis functions we will use for the Finite Element Method in one dimension are piecewise linear functions. To construct piecewise linear functions in *Mathematica*, we use the Piecewise construct. In this construct we divide a function up into two are more segments. For each segment we specify a function value and a range of values for the independent variable where that value applies.

Our goal is to produce a function that takes the form of a 'spike' of height 1 centered at x = c with support on the interval (c-h,c+h). The following code gives us the function we want.

In[ ]:=
```
Spike[c_, h_, x_] := Piecewise[
  {{1 + (x - c) / h, c - h ≤ x && x < c}, {1 - (x - c) / h, c ≤ x && x < c + h}, {0, True}}]
```

In[ ]:=
```
Plot[Spike[1, 0.5, x], {x, 0, 2}]
```

Out[ ]=



Since the Piecewise function plays well with operations such as integration, we can easily integrate functions defined in terms of piecewise functions.

In[ ]:=
```
Integrate[Spike[1, 0.5, x], {x, 0, 2}]
```

Out[ ]=
```
0.5
```

Setting up the Rayleigh-Ritz method calculation

To construct our set of basis functions, we imagine dividing the interval [0,1] into a regular grid of points with the following spacing.

In[ ]:=
```
h[n_] := 1 / (n + 1)
```

The spikes are centered at points ct[i], where i ranges from 1 to n.

In[ ]:=
```
ct[i_] := i h[NN]
```

Note that we have to be careful in setting up these spike functions to ensure that they all vanish at x = 0 and at x = 1. If you look carefully at the way that h and c are defined above, you will see that the first spike just vanishes at x = 0, and the last spike at n = NN just vanishes at x = 1. All the other spikes have their support away from the end points, so we don't have to worry about them vanishing properly at x =

0 and x = 1.

```
In[•]:= phi[i_][x_] := Spike[ct[i], h[NN], x]
```

The next two definitions set up the integrals on the left and right hand side of our system of equations.

```
In[•]:= a[f_, g_] := Integrate[p[x] × D[f[x], x] × D[g[x], x] + q[x] × f[x] × g[x], {x, 0, 1}]
```

```
In[•]:= rhs[f_, g_] := Integrate[f[x] × g[x], {x, 0, 1}]
```

For our first example we will use a basis set of 10 spike functions.

```
In[•]:= NN = 10
Out[•]=
10
```

Here are the p(x), q(x), and f(x) for our example BVP.

```
In[•]:= p[x_] = 1 + x
Out[•]=
1 + x
```

```
In[•]:= q[x_] = 0
Out[•]=
0
```

```
In[•]:= f[x_] = 1
Out[•]=
1
```

Everything is in place for us to construct the linear system. The next two commands compute the matrix A and the right hand side vector b.

```
In[•]:= A = Table[a[phi[j], phi[i]], {i, 1, NN}, {j, 1, NN}]
Out[•]=
{{24, -25/2, 0, 0, 0, 0, 0, 0, 0, 0}, {-25/2, 26, -27/2, 0, 0, 0, 0, 0, 0, 0},
 {0, -27/2, 28, -29/2, 0, 0, 0, 0, 0, 0}, {0, 0, -29/2, 30, -31/2, 0, 0, 0, 0, 0},
 {0, 0, 0, -31/2, 32, -33/2, 0, 0, 0, 0}, {0, 0, 0, 0, -33/2, 34, -35/2, 0, 0, 0},
 {0, 0, 0, 0, 0, -35/2, 36, -37/2, 0, 0}, {0, 0, 0, 0, 0, 0, -37/2, 38, -39/2, 0},
 {0, 0, 0, 0, 0, 0, 0, -39/2, 40, -41/2}, {0, 0, 0, 0, 0, 0, 0, 0, -41/2, 42}}
```

In[•]:=
```
MatrixForm[A]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 24 & -\frac{25}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{25}{2} & 26 & -\frac{27}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{27}{2} & 28 & -\frac{29}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{29}{2} & 30 & -\frac{31}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{31}{2} & 32 & -\frac{33}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{33}{2} & 34 & -\frac{35}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{35}{2} & 36 & -\frac{37}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{37}{2} & 38 & -\frac{39}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{39}{2} & 40 & -\frac{41}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{41}{2} & 42 \end{pmatrix}$$

In[•]:=
```
b = Table[rhs[f, phi[i]], {i, 1, NN}]
```

Out[•]=

$$\left\{\frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}\right\}$$

There are several methods we can use to solve the resulting system of linear equations. Using LinearSolve is a quick and dirty method of getting a solution. For larger systems we may ultimately want to use a more sophisticated and efficient method.

In[•]:=
```
c = LinearSolve[A, b]
```

Out[•]=

$$\left\{\frac{120\,129\,075\,078\,266}{3\,473\,009\,804\,704\,649}, \frac{205\,389\,571\,025\,146}{3\,473\,009\,804\,704\,649}, \frac{260\,947\,203\,267\,512}{3\,473\,009\,804\,704\,649}, \right.$$
$$\frac{290\,898\,918\,523\,228}{3\,473\,009\,804\,704\,649}, \frac{298\,548\,706\,080\,894}{3\,473\,009\,804\,704\,649}, \frac{286\,599\,829\,903\,910}{3\,473\,009\,804\,704\,649},$$
$$\left.\frac{257\,292\,137\,276\,236}{3\,473\,009\,804\,704\,649}, \frac{212\,502\,257\,003\,352}{3\,473\,009\,804\,704\,649}, \frac{153\,818\,105\,766\,818}{3\,473\,009\,804\,704\,649}, \frac{82\,595\,222\,197\,234}{3\,473\,009\,804\,704\,649}\right\}$$

The approximate solution is a linear combination of the spike functions using these c's as coefficients.

In[•]:=
```
soln[x_] = c.Table[phi[i][x], {i, 1, NN}]
```

Out[•]=

$$\frac{82\,595\,222\,197\,234 \begin{pmatrix} \begin{cases} 1 + 11\left(-\frac{10}{11} + x\right) & \frac{9}{11} \le x \,\&\&\, x < \frac{10}{11} \\ 1 - 11\left(-\frac{10}{11} + x\right) & \frac{10}{11} \le x \,\&\&\, x < 1 \\ 0 & \text{True} \end{cases} \end{pmatrix}}{3\,473\,009\,804\,704\,649} +$$

$$\frac{153\,818\,105\,766\,818 \begin{pmatrix} \begin{cases} 1 + 11\left(-\frac{9}{11} + x\right) & \frac{8}{11} \le x \,\&\&\, x < \frac{9}{11} \\ 1 - 11\left(-\frac{9}{11} + x\right) & \frac{9}{11} \le x \,\&\&\, x < \frac{10}{11} \\ 0 & \text{True} \end{cases} \end{pmatrix}}{3\,473\,009\,804\,704\,649} +$$

$$\frac{212\,502\,257\,003\,352}{3\,473\,009\,804\,704\,649}\begin{pmatrix}\begin{cases}1+11\left(-\frac{8}{11}+x\right) & \frac{7}{11}\le x\,\&\&\,x<\frac{8}{11}\\ 1-11\left(-\frac{8}{11}+x\right) & \frac{8}{11}\le x\,\&\&\,x<\frac{9}{11}\\ 0 & \text{True}\end{cases}\end{pmatrix}+$$

$$\frac{257\,292\,137\,276\,236}{3\,473\,009\,804\,704\,649}\begin{pmatrix}\begin{cases}1+11\left(-\frac{7}{11}+x\right) & \frac{6}{11}\le x\,\&\&\,x<\frac{7}{11}\\ 1-11\left(-\frac{7}{11}+x\right) & \frac{7}{11}\le x\,\&\&\,x<\frac{8}{11}\\ 0 & \text{True}\end{cases}\end{pmatrix}+$$

$$\frac{286\,599\,829\,903\,910}{3\,473\,009\,804\,704\,649}\begin{pmatrix}\begin{cases}1+11\left(-\frac{6}{11}+x\right) & \frac{5}{11}\le x\,\&\&\,x<\frac{6}{11}\\ 1-11\left(-\frac{6}{11}+x\right) & \frac{6}{11}\le x\,\&\&\,x<\frac{7}{11}\\ 0 & \text{True}\end{cases}\end{pmatrix}+$$

$$\frac{298\,548\,706\,080\,894}{3\,473\,009\,804\,704\,649}\begin{pmatrix}\begin{cases}1+11\left(-\frac{5}{11}+x\right) & \frac{4}{11}\le x\,\&\&\,x<\frac{5}{11}\\ 1-11\left(-\frac{5}{11}+x\right) & \frac{5}{11}\le x\,\&\&\,x<\frac{6}{11}\\ 0 & \text{True}\end{cases}\end{pmatrix}+$$

$$\frac{290\,898\,918\,523\,228}{3\,473\,009\,804\,704\,649}\begin{pmatrix}\begin{cases}1+11\left(-\frac{4}{11}+x\right) & \frac{3}{11}\le x\,\&\&\,x<\frac{4}{11}\\ 1-11\left(-\frac{4}{11}+x\right) & \frac{4}{11}\le x\,\&\&\,x<\frac{5}{11}\\ 0 & \text{True}\end{cases}\end{pmatrix}+$$

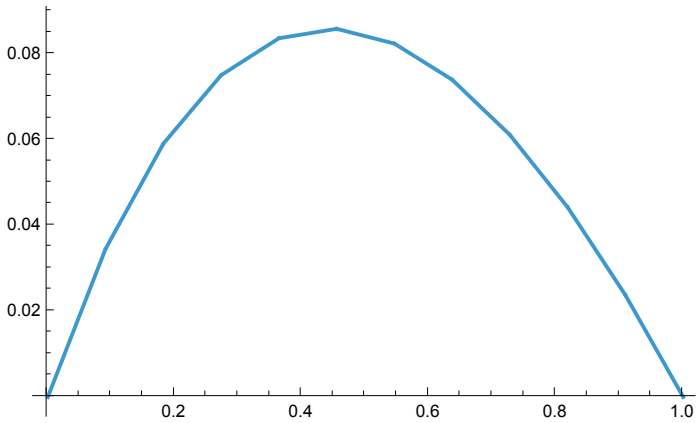
$$\frac{260\,947\,203\,267\,512}{3\,473\,009\,804\,704\,649}\begin{pmatrix}\begin{cases}1+11\left(-\frac{3}{11}+x\right) & \frac{2}{11}\le x\,\&\&\,x<\frac{3}{11}\\ 1-11\left(-\frac{3}{11}+x\right) & \frac{3}{11}\le x\,\&\&\,x<\frac{4}{11}\\ 0 & \text{True}\end{cases}\end{pmatrix}+$$

$$\frac{205\,389\,571\,025\,146}{3\,473\,009\,804\,704\,649}\begin{pmatrix}\begin{cases}1+11\left(-\frac{2}{11}+x\right) & \frac{1}{11}\le x\,\&\&\,x<\frac{2}{11}\\ 1-11\left(-\frac{2}{11}+x\right) & \frac{2}{11}\le x\,\&\&\,x<\frac{3}{11}\\ 0 & \text{True}\end{cases}\end{pmatrix}+$$

$$\frac{120\,129\,075\,078\,266}{3\,473\,009\,804\,704\,649}\begin{pmatrix}\begin{cases}1+11\left(-\frac{1}{11}+x\right) & 0\le x\,\&\&\,x<\frac{1}{11}\\ 1-11\left(-\frac{1}{11}+x\right) & \frac{1}{11}\le x\,\&\&\,x<\frac{2}{11}\\ 0 & \text{True}\end{cases}\end{pmatrix}$$

In[•]:=
```
Plot[soln[x], {x, 0, 1}]
```

Out[ ]=



Mathematica has a powerful function DSolve for solving differential equations. The equation we are working with here is sufficiently simple that DSolve can solve it exactly.

In[ ]:=
```
actual = DSolve[{-D[(p[x] y'[x]), x] == 1, y[0] == 0, y[1] == 0}, y, x]
```

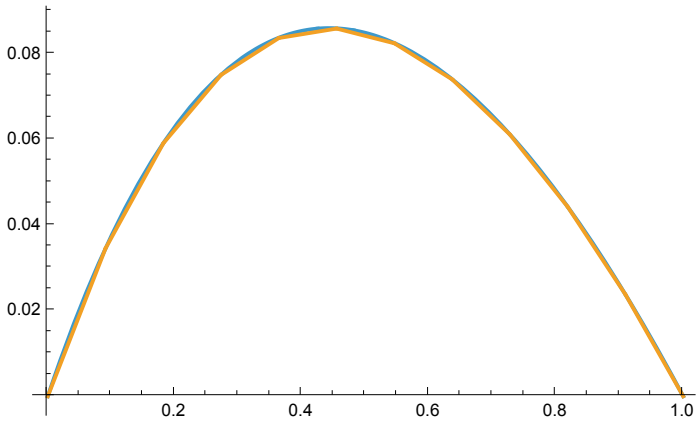
Out[ ]=

$$\left\{\left\{y \to \text{Function}\left[\{x\}, \frac{-x \text{Log}[2] + \text{Log}[1+x]}{\text{Log}[2]}\right]\right\}\right\}$$

Here are plots comparing the actual and approximate solutions and difference between these two solutions.

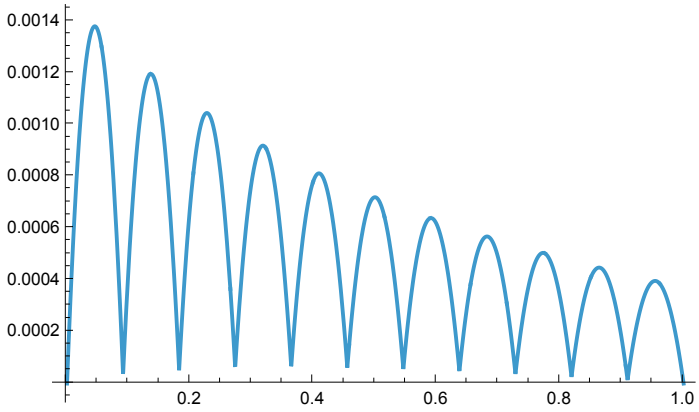
In[ ]:=
```
Plot[{y[x] /. actual, soln[x]}, {x, 0, 1}]
```

Out[ ]=



In[•]:=
```
Plot[(y[x] /. actual[[1]]) - soln[x], {x, 0, 1}]
```

Out[•]=



More sample points...

Solving the same problem with more mesh points and a correspondingly larger basis should produce better approximations.

In[•]:=
```
NN = 50
```

Out[•]=
```
50
```

Increasing the number of basis functions increases the size of the linear system, which in turn increases the amount of time needed to compute the A matrix. *Mathematica*'s Timing function can tell us how long it takes to do a particular calculation.

Since a 50 by 50 matrix would be too big to print, we suppress printing of the matrix A by placing a semicolon at the end of the command to compute the matrix.

In[•]:=
```
Timing[A = Table[a[phi[j], phi[i]], {i, 1, NN}, {j, 1, NN}];]
```

Out[•]=
```
{29.8656, Null}
```

We also suppress printing of the other parts of the calculation.

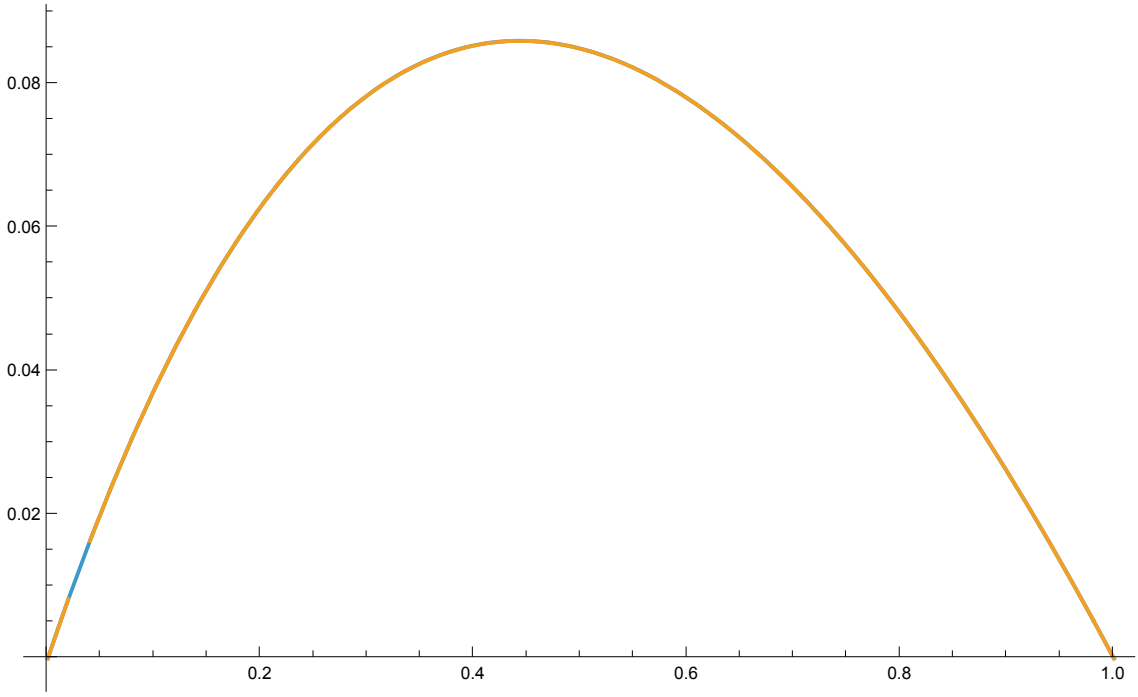
In[•]:=
```
b = Table[rhs[f, phi[i]], {i, 1, NN}];
```

In[•]:=
```
c = LinearSolve[A, b];
```

In[•]:=
```
soln[x_] = c.Table[phi[i][x], {i, 1, NN}];
```

In[•]:=
```
Plot[{y[x] /. actual, soln[x]}, {x, 0, 1}]
```

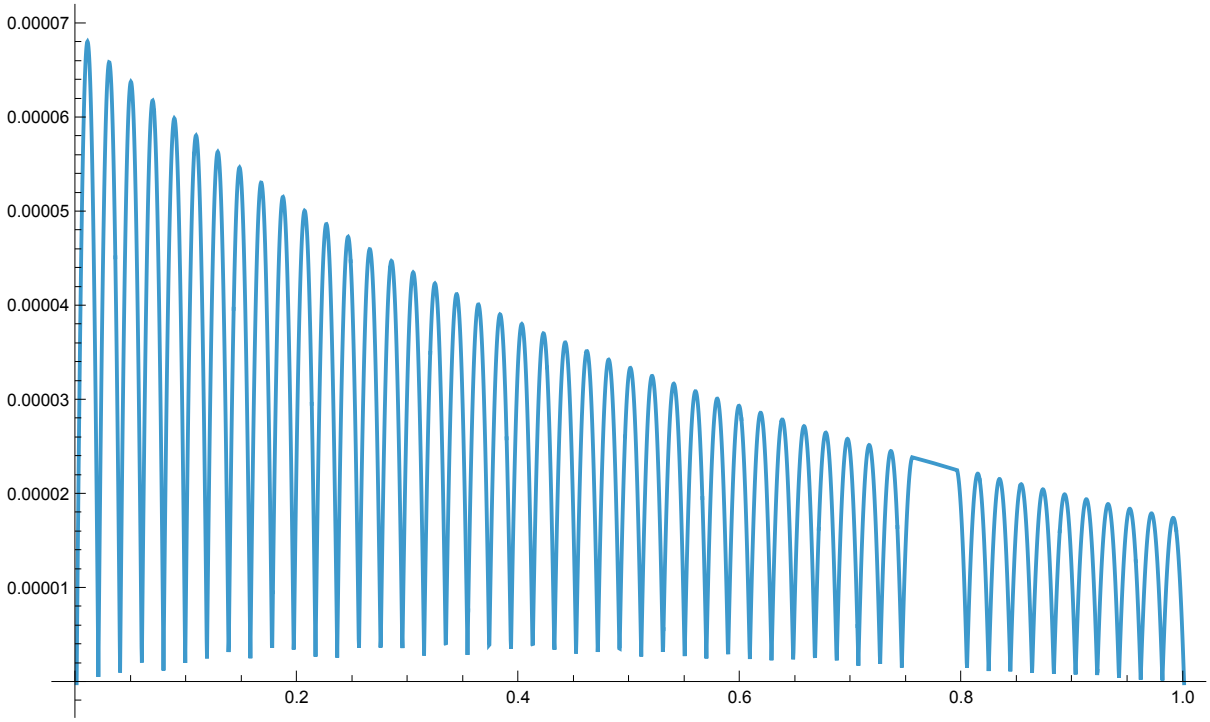
Out[•]=



Increasing the number of sample points by a factor of 2.5 decreases the maximum error by a factor of about 80.

In[•]:=
```
Plot[(y[x] /. actual[[1]]) - soln[x], {x, 0, 1}]
```

Out[•]=



A harder problem

To make a more challenging problem we can increase the complexity of the p(x) and f(x) functions.

```
In[•]:= NN = 20
Out[•]=
20

In[•]:= p[x_] = 1 + x^2
Out[•]=
1 + x^2

In[•]:= f[x_] = Exp[x]
Out[•]=
e^x

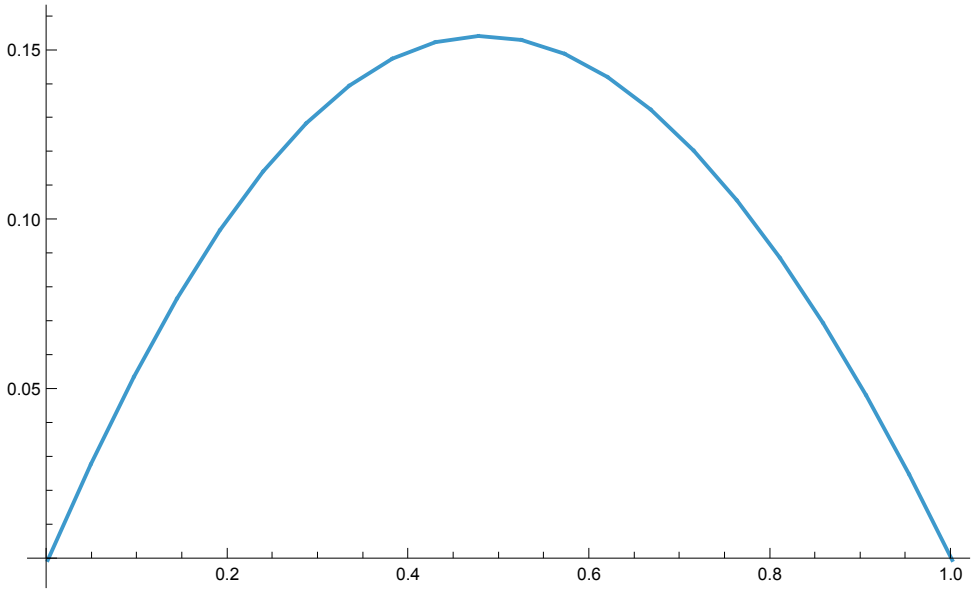
In[•]:= Timing[A = Table[a[phi[j], phi[i]], {i, 1, NN}, {j, 1, NN}];]
Out[•]=
{4.64856, Null}

In[•]:= b = Table[rhs[f, phi[i]], {i, 1, NN}];

In[•]:= c = LinearSolve[A, b];

In[•]:= soln[x_] = c.Table[phi[i][x], {i, 1, NN}];

In[•]:= Plot[soln[x], {x, 0, 1}]
Out[•]=
```



Mathematica is still able to solve the BVP exactly.

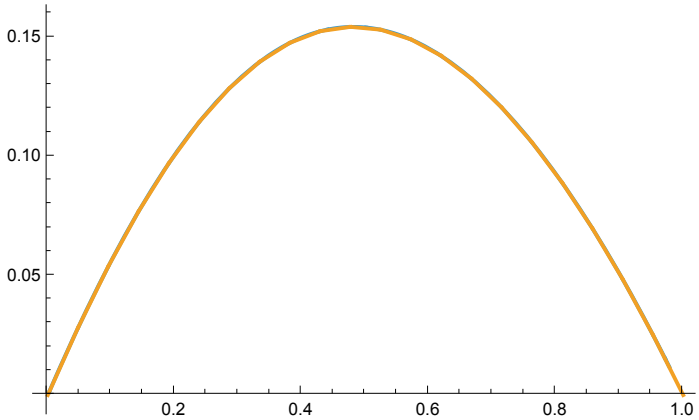
In[•]:=
```
actual = DSolve[{-D[(p[x] × y'[x]), x] == f[x], y[0] == 0, y[1] == 0}, y, x]
```

Out[•]=

$$\left\{\left\{y \to \text{Function}\left[\{x\},\right.\right.\right.$$
$$-\frac{1}{2\pi}\, i\, e^{-i}\left(e^{2i}\pi\,\text{ExpIntegralEi}[-i] - 4\,e^{2i}\,\text{ArcTan}[x]\,\text{ExpIntegralEi}[-i] -\right.$$
$$\pi\,\text{ExpIntegralEi}[i] + 4\,\text{ArcTan}[x]\,\text{ExpIntegralEi}[i] +$$
$$4\,e^{2i}\,\text{ArcTan}[x]\,\text{ExpIntegralEi}[1-i] - 4\,\text{ArcTan}[x]\,\text{ExpIntegralEi}[1+i] -$$
$$\left.\left.\left.e^{2i}\pi\,\text{ExpIntegralEi}[-i+x] + \pi\,\text{ExpIntegralEi}[i+x]\right)\right]\right\}\right\}$$

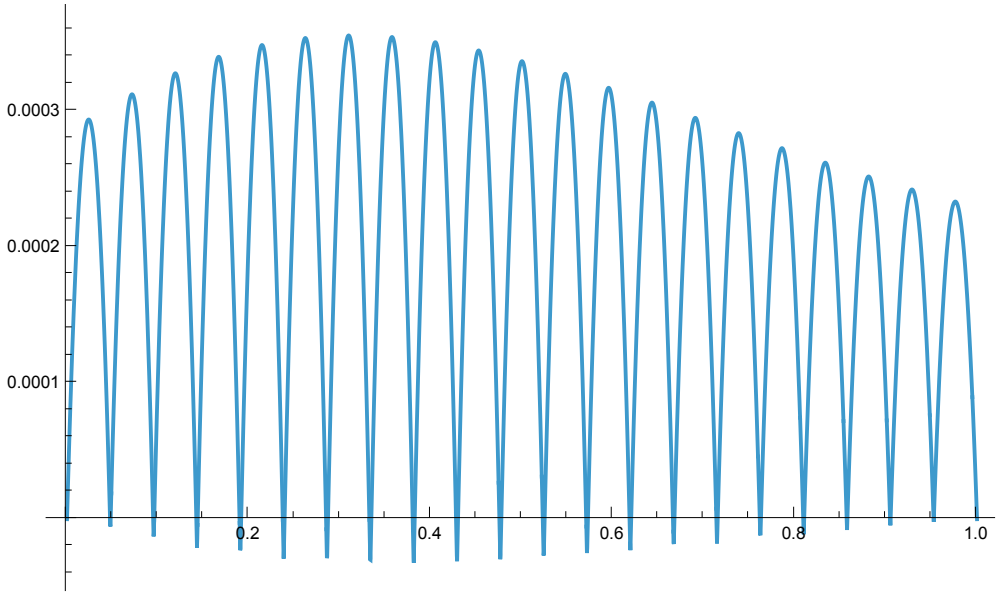
In[•]:=
```
Plot[{y[x] /. actual, soln[x]}, {x, 0, 1}]
```

Out[•]=



In[•]:=
```
Plot[Re[(y[x] /. actual[[1]])] - soln[x], {x, 0, 1}]
```

Out[•]=



Making the calculations run faster

The most expensive part of the calculations needed for the finite element method is the computation of the matrix K. The following calculations explore a strategy for speeding up that calculation.

What we are going to do is to use *Mathematica*'s symbolic calculation capabilities and a little bit of common sense to pre-compute a formula for the elements of the matrix K. To start the process, we clear the definition of the variable NN, which determines the step sizes to use and the number of basis functions. The result is that the formula for the basis functions will contain NN as a variable.

```
In[•]:= NN =.
```

```
In[•]:= phi[j][x]
```

Out[•]=

$$\begin{cases} 1+(1+\text{NN})\left(-\frac{j}{1+\text{NN}}+x\right) & -\frac{1}{1+\text{NN}}+\frac{j}{1+\text{NN}} \le x \;\&\&\; x < \frac{j}{1+\text{NN}} \\ 1-(1+\text{NN})\left(-\frac{j}{1+\text{NN}}+x\right) & \frac{j}{1+\text{NN}} \le x \;\&\&\; x < \frac{1}{1+\text{NN}}+\frac{j}{1+\text{NN}} \\ 0 & \text{True} \end{cases}$$

Of course we can get *Mathematica* to compute the derivative of this function.

```
In[•]:= D[phi[i][x], x]
```

Out[•]=

$$\begin{cases} 1+\text{NN} & \frac{1}{1+\text{NN}}-\frac{i}{1+\text{NN}}+x \ge 0 \;\&\&\; \frac{i}{1+\text{NN}}-x > 0 \\ -1-\text{NN} & \frac{i}{1+\text{NN}}-x \le 0 \;\&\&\; \frac{1}{1+\text{NN}}+\frac{i}{1+\text{NN}}-x > 0 \\ 0 & \text{True} \end{cases}$$

Because the basis functions are so simple in form, we can potentially save a tiny bit of work by computing their derivatives by hand and simply programming that into Mathematica.

```
In[•]:= dPhi[i_][x_] := Piecewise[{{1 / h[NN], ct[i] - h[NN] ≤ x && x < ct[i]},
    {-1 / h[NN], ct[i] ≤ x && x < ct[i] + h[NN]}, {0, True}}]
```

This produces the same result as before.

```
In[•]:= dPhi[i][x]
```

Out[•]=

$$\begin{cases} 1+\text{NN} & -\frac{1}{1+\text{NN}}+\frac{i}{1+\text{NN}} \le x \;\&\&\; x < \frac{i}{1+\text{NN}} \\ -1-\text{NN} & \frac{i}{1+\text{NN}} \le x \;\&\&\; x < \frac{1}{1+\text{NN}}+\frac{i}{1+\text{NN}} \\ 0 & \text{True} \end{cases}$$

We now have everything in place to compute a symbolic expression for A[i,j].

```
In[•]:= NN = 20
```

Out[•]=

```
20
```

```
In[•]:= p[x_] = 1 + x^2
```

Out[•]=

$$1+x^2$$

```
In[•]:= f[x_] = Exp[x]
```

Out[•]=

$$e^x$$

Before we compute the matrix we have to adjust the definition of a(u,v) because we are going to be

supplying our own derivatives.

```
In[•]:= a[f_, g_] := Integrate[p[x] × f[x] × g[x], {x, 0, 1}]
```

We now compute a (rather complicated) symbolic expression for the entries of the A matrix.

```
In[•]:= AA[i_, j_] = a[dPhi[j], dPhi[i]]
```

```
Out[•]=
```

$$\begin{cases}
-\frac{190}{9} & i == 1 \&\& j == 2 \\
-\frac{1324}{63} + 21\,j + \frac{j^3}{63} & i == 1 \&\& 0 < j < 1 \\
\frac{1324}{63} - 21\,j - \frac{j^3}{63} & i == 1 \&\& 1 < j < 2 \\
-\frac{442}{7} + \frac{442\,j}{21} - \frac{j^2}{21} + \frac{j^3}{63} & i == 1 \&\& 2 < j < 3 \\
-\frac{352}{9} & i == 20 \&\& j == 19 \\
\frac{3408}{7} - \frac{442\,j}{21} - \frac{j^2}{21} - \frac{j^3}{63} & i == 20 \&\& 18 < j < 19 \\
-\frac{34\,460}{63} + 21\,j + \frac{j^3}{63} & i == 20 \&\& 19 < j < 20 \\
\frac{34\,460}{63} - 21\,j - \frac{j^3}{63} & i == 20 \&\& 20 < j < 21 \\
-21\,j - \frac{j^3}{63} & -1 < i \leq 0 \&\& 0 < j \leq 1 + i \\
-21 - 21\,i - \frac{1}{63}\,(1+i)^3 & -1 < i \leq 0 \&\& 1 + i < j \leq 1 \\
-21\,i - \frac{i^3}{63} & 0 < i < 1 \&\& -1 + i < j \leq 0 \\
-588 + 21\,i + \frac{i^3}{63} & 20 < i < 21 \&\& 21 < j < 1 + i \\
-\frac{38\,368}{63} + \frac{442\,i}{21} - \frac{i^2}{21} + \frac{i^3}{63} & 21 < i < 22 \&\& 20 < j < -1 + i \\
-21 - 21\,j - \frac{1}{63}\,(1+j)^3 & (i == 1 \&\& -1 < j \leq 0) \,||\, (0 < i < 1 \&\& -1 < j \leq -1 + i) \\
-\frac{38\,368}{63} + \frac{442\,j}{21} - \frac{j^2}{21} + \frac{j^3}{63} & (i == 20 \&\& 21 \leq j < 22) \,||\, (20 < i < 21 \&\& 1 + i \leq j < 22) \\
-588 + 21\,j + \frac{j^3}{63} & (i == 21 \&\& -1 + i \leq j < 21) \,||\, (21 < i < 22 \&\& -1 + i \leq j < 21) \\
-\frac{1324}{63} + \frac{j}{21} - \frac{j^2}{21} & (0 < i < 1 \&\& j == 1 + i) \,||\, (1 < i < 20 \&\& j == 1 + i) \\
-\frac{1324}{63} - \frac{j}{21} - \frac{j^2}{21} & (1 < i < 20 \&\& j == -1 + i) \,||\, (20 < i < 21 \&\& j == -1 + i) \\
-\frac{2648}{63} - \frac{442\,i}{21} - \frac{i^2}{21} - \frac{i^3}{63} + \frac{442\,j}{21} - \frac{j^2}{21} + \frac{j^3}{63} & (-1 < i \leq 0 \&\& 1 < j < 2 + i) \,||\, (0 < i < 1 \&\& 1 + i < j < 2 + i) \,||\, (1 < i < 20 \&\& 1 + i < j < 2 + i) \\
-21\,i - \frac{i^3}{63} + 21\,j + \frac{j^3}{63} & (0 < i < 1 \&\& 0 < j < i) \,||\, (1 < i < 20 \&\& -1 + i < j < i) \,||\, (20 < i < 21 \&\& -1 + i < j < i) \\
21\,i + \frac{i^3}{63} - 21\,j - \frac{j^3}{63} & (0 < i < 1 \&\& i < j < 1 + i) \,||\, (1 < i < 20 \&\& i < j < 1 + i) \,||\, (20 < i < 21 \&\& i < j \leq 21) \\
-\frac{2648}{63} + \frac{442\,i}{21} - \frac{i^2}{21} + \frac{i^3}{63} - \frac{442\,j}{21} - \frac{j^2}{21} - \frac{j^3}{63} & (i == 21 \&\& -2 + i < j < -1 + i) \,||\, (1 < i < 20 \&\& -2 + i < j < -1 + i) \,||\, (20 < i < 21 \&\& -2 + i < j < -1 + i) \,||\, (21 < i < 22 \&\& -2 + i < j \leq 20) \\
0 & \text{True}
\end{cases}$$

Although this expression is complex, it is easier to evaluate than an integral. This greatly speeds up the

computation of the matrix A: what previously took more than 4 seconds now only takes a fraction of a second.

```
In[•]:= Timing[A = Table[AA[i, j], {i, 1, NN}, {j, 1, NN}];]
Out[•]=
{0.018483, Null}
```

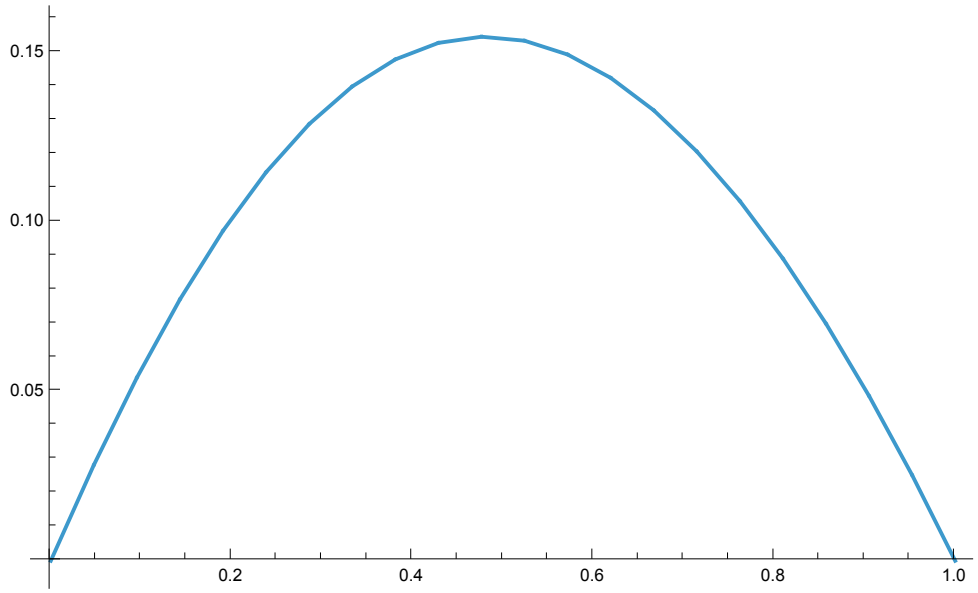
```
In[•]:= b = Table[rhs[f, phi[i]], {i, 1, NN}];
```

```
In[•]:= c = LinearSolve[A, b];
```

```
In[•]:= soln[x_] = c.Table[phi[i][x], {i, 1, NN}];
```

The solution produced by this method is identical to that produced above.

```
In[•]:= Plot[soln[x], {x, 0, 1}]
Out[•]=
```




```
In[•]:= actual = DSolve[{-D[(p[x] × y'[x]), x] == f[x], y[0] == 0, y[1] == 0}, y, x]
Out[•]=
```

$$\left\{\left\{y \to \text{Function}\left[\{x\},\right.\right.\right.$$

$$-\frac{1}{2\,\pi}\,i\,e^{-i}\left(e^{2\,i}\,\pi\,\text{ExpIntegralEi}[-i] - 4\,e^{2\,i}\,\text{ArcTan}[x]\,\text{ExpIntegralEi}[-i] - \right.$$

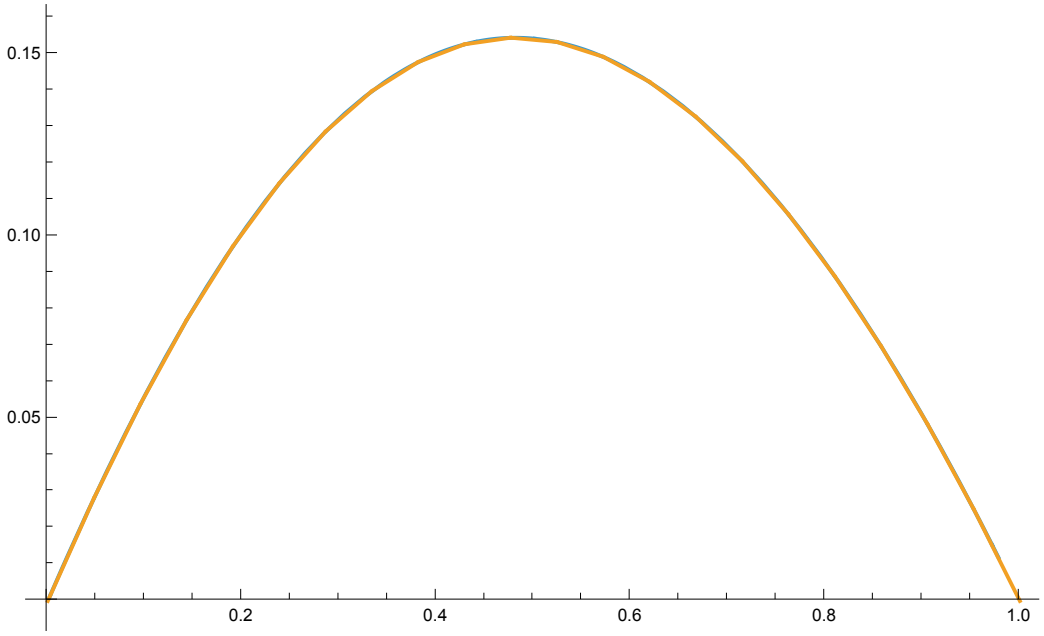
$$\pi\,\text{ExpIntegralEi}[i] + 4\,\text{ArcTan}[x]\,\text{ExpIntegralEi}[i] +$$

$$4\,e^{2\,i}\,\text{ArcTan}[x]\,\text{ExpIntegralEi}[1-i] - 4\,\text{ArcTan}[x]\,\text{ExpIntegralEi}[1+i] -$$

$$\left.\left.\left.\left.e^{2\,i}\,\pi\,\text{ExpIntegralEi}[-i+x] + \pi\,\text{ExpIntegralEi}[i+x]\right)\right]\right\}\right\}$$

In[•]:=
```
Plot[{y[x] /. actual, soln[x]}, {x, 0, 1}]
```

Out[•]=



In[•]:=
```
Plot[Re[(y[x] /. actual[[1]])] - soln[x], {x, 0, 1}]
```

Out[•]=

