The weak form of a boundary value problem

A special case of the general boundary value problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right)+q(x)\,y=f(x)$$

$$f(a)=\alpha$$

$$f(b)=\beta$$

is the Dirichlet BVP

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right)+q(x)\,y=f(x)$$

$$f(a)=0$$

$$f(b)=0$$

One method for solving the Dirichlet BVP converts the differential equation into its so-called weak form:

$$I[y]=\int_a^b p(x)\,(y'(x))^2+q(x)\,y(x)^2-2f(x)\,y(x)\,\mathrm{d}x$$

$y(x)$ satisfies the boundary value problem if and only if $y(x)$ minimizes $I[y]$.

The Rayleigh-Ritz method

To minimize I[y] we have to let y range of all twice-differentiable functions in the space $C_D^2[a,b]$ that vanish at the boundary. We can obtain an approximate solution by instead letting y range over all functions $\varphi(x)$ in some more tractable, finite dimensional subspace of $C_D^2[a,b]$.

To construct our candidate functions $\varphi(x)$ we form a linear combination of *basis functions* $\varphi_i(x)$. The basis functions are typically chosen to satisfy the boundary conditions and to be *approximately orthogonal*.

We then seek

$$\varphi(x)=\sum_{i=1}^{n} c_i\,\varphi_i(x)$$

that minimizes $I[\varphi]$. To find the minimum we demand that

$$\frac{\partial}{\partial c_i}I[\varphi]=0$$

for all i. This ends up giving us a system of linear equations that the coefficients c_i must satisfy:

$$a_{i,j}=\int_a^b p(x)\,\varphi_i'(x)\,\varphi_j'(x)+q(x)\,\varphi_i(x)\,\varphi_j(x)\;\mathrm{d}x$$

$$b_i = \int_a^b f(x)\, \varphi_i(x)\, \mathrm{d}x$$

$$A\, \vec{c} = \vec{b}$$

The accompanying Mathematica notebook will show how this method works in practice for a series of examples.