Making better polynomials

Basis polynomials and the Lagrange interpolation scheme make a good place to start the discussion of interpolating polynomials, but ultimately this approach is quite clunky. The scheme this produces for constructing interpolating polynomials

$$L(x) = \sum_{k=0}^{n} y_k \, L_{n,k}(x)$$

is quite messy to work with. We have already seen that one way to bypass complications caused by this messy scheme is to use a recursive approach, Neville's method, to evaluate the polynomials without actually constructing them.

Sometimes, however, you just have to have actual polynomials. In section 3.3 we are going to encounter an alternative scheme for constructing interpolating polynomials called the Newton scheme. This scheme has the advantage of being easier to do.

The Newton method starts by assuming that the interpolating polynomial takes a particular form which is simpler than the form the Lagrange polynomials are based on. Here is one example of a Newton form polynomial designed to interpolate the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, ... , $(x_n, f(x_n))$:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

Provided we can come up with a convenient method for computing the coefficients, this method has one big advantage for constructing polynomials symbolically. Since each successive polynomial is the previous polynomial multiplied by an additional linear factor, computing the individual polynomials that make up this sum symbolically is a much easier task than computing a collection of Lagrange basis polynomials. In fact, for modest values of n it is not too hard to compute these polynomials by hand.

Computing the coefficients of the Newton polynomial by brute force

The Newton interpolating polynomial for the data points (x_0, y_0), (x_1, y_1), …, (x_n, y_n) takes the form

$$p(x) = a_0 + a_1 \left(x - x_0\right) + a_2 \left(x - x_0\right) \left(x - x_1\right) + \cdots + a_n \left(x - x_0\right)\cdots\left(x - x_{n-1}\right)$$

Because this an interpolating polynomial it must satisfy the equations

$$p\left(x_0\right) = y_0$$

$$p\left(x_1\right) = y_1$$

$$\vdots$$

$$p\left(x_n\right) = y_n$$

To begin to see whether or not there is a formula for the unknown coefficients a_0 , a_1 , … , a_n we will start by working with a small example. Suppose we needed to construct a Newton polynomial that interpolates a set of four data points (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3). That polynomial would have the structure

$$p(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)(x - x_1) + a_3 (x - x_0)(x - x_1)(x - x_2)$$

and would have to satisfy the equations

$$p(x_0) = y_0$$

$$p(x_1) = y_1$$

$$p(x_2) = y_2$$

$$p(x_3) = y_3$$

or

$$a_0 = y_0$$

$$a_0 + a_1 (x_1 - x_0) = y_1$$

$$a_0 + a_1 (x_2 - x_0) + a_2 (x_2 - x_0)(x_2 - x_1) = y_2$$

$$a_0 + a_1 (x_3 - x_0) + a_2 (x_3 - x_0)(x_3 - x_1) + a_3 (x_3 - x_0)(x_3 - x_1)(x_3 - x_2) = y_3$$

This set of equations can be viewed as a linear system:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) & 0 \\ 1 & (x_3 - x_0) & (x_3 - x_0)(x_3 - x_1) & (x_3 - x_0)(x_3 - x_1)(x_3 - x_2) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

This system can be solved by forward substitution: here are the first two equations:

$$a_0 = y_0$$

$$a_0 + a_1 (x_1 - x_0) = y_0 + a_1 (x_1 - x_0) = y_1$$

Solving the second of these equations for a_1 yields

$$a_1 = \frac{y_1 - y_0}{x_1 - x_0}$$

For reasons that will become clear in a moment, it is better to write this solution as

$$a_1 = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0}$$

The next equation is

$$a_0 + a_1 \left(x_2 - x_0\right) + a_2 \left(x_2 - x_0\right) \left(x_2 - x_1\right) = y_2$$

or

$$a_2 = \frac{y_2 - \left(\frac{y_1}{x_1 - x_0} + \frac{y_0}{x_0 - x_1}\right) \left(x_2 - x_0\right) - y_0}{\left(x_2 - x_0\right) \left(x_2 - x_1\right)}$$

$$a_2 = \frac{y_0 + \frac{y_0}{x_0 - x_1} \left(x_2 - x_0\right)}{\left(x_0 - x_2\right) \left(x_2 - x_1\right)} + \frac{y_1}{\left(x_1 - x_0\right) \left(x_1 - x_2\right)} + \frac{y_2}{\left(x_2 - x_0\right) \left(x_2 - x_1\right)}$$

The first term can be rewritten

$$\frac{y_0 + \frac{y_0}{x_0 - x_1} \left(x_2 - x_0\right)}{\left(x_0 - x_2\right) \left(x_2 - x_1\right)} = \frac{y_0 \frac{x_2 - x_1}{x_0 - x_1}}{\left(x_0 - x_2\right) \left(x_2 - x_1\right)} = \frac{y_0}{\left(x_0 - x_1\right) \left(x_0 - x_2\right)}$$

This gives us that

$$a_2 = \frac{y_0}{\left(x_0 - x_1\right) \left(x_0 - x_2\right)} + \frac{y_1}{\left(x_1 - x_0\right) \left(x_1 - x_2\right)} + \frac{y_2}{\left(x_2 - x_0\right) \left(x_2 - x_1\right)}$$

We have seen enough now to form a conjecture about the last coefficient:

$$a_3 = \frac{y_0}{\left(x_0 - x_1\right) \left(x_0 - x_2\right) \left(x_0 - x_3\right)} + \frac{y_1}{\left(x_1 - x_0\right) \left(x_1 - x_2\right) \left(x_1 - x_3\right)}$$
$$+ \frac{y_2}{\left(x_2 - x_0\right) \left(x_2 - x_1\right) \left(x_2 - x_3\right)} + \frac{y_3}{\left(x_3 - x_0\right) \left(x_3 - x_1\right) \left(x_3 - x_2\right)}$$

Let us now confirm that this is true by plugging in all four coefficients in the last equation:

$$y_0 + \left(\frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0}\right) \left(x_3 - x_0\right) + \left(\frac{y_0}{\left(x_0 - x_1\right) \left(x_0 - x_2\right)} + \frac{y_1}{\left(x_1 - x_0\right) \left(x_1 - x_2\right)} + \frac{y_2}{\left(x_2 - x_0\right) \left(x_2 - x_1\right)}\right) \left(x_3 - x_0\right)$$
$$\left(x_3 - x_1\right) + (\frac{y_0}{\left(x_0 - x_1\right) \left(x_0 - x_2\right) \left(x_0 - x_3\right)} + \frac{y_1}{\left(x_1 - x_0\right) \left(x_1 - x_2\right) \left(x_1 - x_3\right)} + \frac{y_2}{\left(x_2 - x_0\right) \left(x_2 - x_1\right) \left(x_2 - x_3\right)} +$$
$$\frac{y_3}{\left(x_3 - x_0\right) \left(x_3 - x_1\right) \left(x_3 - x_2\right)}) \left(x_3 - x_0\right) \left(x_3 - x_1\right) \left(x_3 - x_2\right)$$

The next step is to reorganize all of the terms here to collect terms that have the same factors y_k:

$$y_3 \frac{\left(x_3 - x_0\right) \left(x_3 - x_1\right) \left(x_3 - x_2\right)}{\left(x_3 - x_0\right) \left(x_3 - x_1\right) \left(x_3 - x_2\right)}$$

$$+\, y_2\left(\frac{(x_3 - x_0)\,(x_3 - x_1)\,(x_3 - x_2)}{(x_2 - x_0)\,(x_2 - x_1)\,(x_2 - x_3)} + \frac{(x_3 - x_0)\,(x_3 - x_1)}{(x_2 - x_0)\,(x_2 - x_1)}\right)$$

$$+\, y_1\left(\frac{(x_3 - x_0)\,(x_3 - x_1)\,(x_3 - x_2)}{(x_1 - x_0)\,(x_1 - x_2)\,(x_1 - x_3)} + \frac{(x_3 - x_0)\,(x_3 - x_1)}{(x_1 - x_0)\,(x_1 - x_2)} + \frac{x_3 - x_0}{x_1 - x_0}\right)$$

$$+\, y_0\left(\frac{(x_3 - x_0)\,(x_3 - x_1)\,(x_3 - x_2)}{(x_0 - x_1)\,(x_0 - x_2)\,(x_0 - x_3)} + \frac{(x_3 - x_0)\,(x_3 - x_1)}{(x_0 - x_1)\,(x_0 - x_2)} + \frac{x_3 - x_0}{x_0 - x_1} + 1\right)$$

The first term clearly simplifies to y_3. By getting common denominators and combining terms we can show that the quantities in parentheses in the remaining terms all simplify to 0:

$$\frac{(x_3 - x_0)\,(x_3 - x_1)\,(x_3 - x_2)}{(x_2 - x_0)\,(x_2 - x_1)\,(x_2 - x_3)} + \frac{(x_3 - x_0)\,(x_3 - x_1)}{(x_2 - x_0)\,(x_2 - x_1)}$$

$$= \frac{(x_3 - x_0)\,(x_3 - x_1)\,(x_3 - x_2)}{(x_2 - x_0)\,(x_2 - x_1)\,(x_2 - x_3)} + \frac{(x_3 - x_0)\,(x_3 - x_1)(x_2 - x_3)}{(x_2 - x_0)\,(x_2 - x_1)\,(x_2 - x_3)} = 0$$

$$\frac{(x_3 - x_0)\,(x_3 - x_1)\,(x_3 - x_2)}{(x_1 - x_0)\,(x_1 - x_2)\,(x_1 - x_3)} + \frac{(x_3 - x_0)\,(x_3 - x_1)}{(x_1 - x_0)\,(x_1 - x_2)} + \frac{x_3 - x_0}{x_1 - x_0}$$

$$= \frac{(x_3 - x_0)\,(x_3 - x_1)\,(x_3 - x_2)}{(x_1 - x_0)\,(x_1 - x_2)\,(x_1 - x_3)} + \frac{(x_3 - x_0)\,(x_2 - x_3)}{(x_1 - x_0)\,(x_2 - x_1)}$$

$$= \frac{(x_3 - x_0)\,(x_3 - x_1)\,(x_3 - x_2)}{(x_1 - x_0)\,(x_1 - x_2)\,(x_1 - x_3)} + \frac{(x_3 - x_0)\,(x_3 - x_2)\,(x_1 - x_3)}{(x_1 - x_0)\,(x_1 - x_2)\,(x_1 - x_3)} = 0$$

$$\frac{(x_3 - x_0)\,(x_3 - x_1)\,(x_3 - x_2)}{(x_0 - x_1)\,(x_0 - x_2)\,(x_0 - x_3)} + \frac{(x_3 - x_0)\,(x_3 - x_1)}{(x_0 - x_1)\,(x_0 - x_2)} + \frac{x_3 - x_0}{x_0 - x_1} + 1$$

$$= \frac{(x_3 - x_0)\,(x_3 - x_1)\,(x_3 - x_2)}{(x_0 - x_1)\,(x_0 - x_2)\,(x_0 - x_3)} + \frac{(x_3 - x_0)\,(x_3 - x_1)}{(x_0 - x_1)\,(x_0 - x_2)} - \frac{x_3 - x_1}{x_1 - x_0}$$

$$= \frac{(x_3 - x_0)\,(x_3 - x_1)\,(x_3 - x_2)}{(x_0 - x_1)\,(x_0 - x_2)\,(x_0 - x_3)} + \frac{(x_3 - x_1)\,(x_3 - x_2)}{(x_1 - x_0)\,(x_2 - x_0)}$$

$$= \frac{(x_3 - x_0)\,(x_3 - x_1)\,(x_3 - x_2)}{(x_0 - x_1)\,(x_0 - x_2)\,(x_0 - x_3)} + \frac{(x_3 - x_1)\,(x_3 - x_2)\,(x_0 - x_3)}{(x_1 - x_0)\,(x_2 - x_0)\,(x_0 - x_3)} = 0$$

Divided differences

From the example above we can see that the coefficients of the Newton polynomial follow a simple and predictable pattern. This pattern can be further simplified by noticing that the denominators in the terms exhibit an obvious regularity. A clever way to capture the pattern in the denominators in the terms that make up a_k is to introduce

$$q_k(x) = \left(x - x_0\right) \left(x - x_1\right) \cdots \left(x - x_k\right)$$

and then write

$$a_k = \sum_{j=0}^{k} \frac{y_j}{q_k{}'(x_j)}$$

The latter form is common known as the *expanded form* of the Newton divided differences.

At this point this form for the coefficients of the Newton polynomial merely has the status of a strong conjecture. To start moving in the direction of an actual proof, we next take a surprising step: we ignore almost everything we have learned about the coefficients and retain only one idea as a conjecture:

$$a_k \text{ depends on the values of } [x_0 , x_1 , \ldots, x_k]$$

To emphasize this fact we rewrite the coefficients of the Newton polynomial:

$$p(x) = f[x_0] + f[x_0 , x_1] \left(x - x_0\right) + f[x_0 , x_1 , x_2] \left(x - x_0\right) \left(x - x_1\right) + \cdots$$

We now demonstrate that these coefficients have a simple recursive structure. Our tool for doing this is the same recursive formula we used to develop Neville's method.

$$p_{i,j}(x) = \frac{(x - x_j)\, p_{i,j-1}(x) + (x_i - x)\, p_{i+1,j}(x)}{x_i - x_j}$$

More specifically, we would like to apply the version of this formula with $i = 0$ and $j = n$:

$$p_{0,n}(x) = \frac{(x - x_n)\, p_{0,n-1}(x) + (x_0 - x)\, p_{1,n}(x)}{x_0 - x_n}$$

This recursive relation applies to all interpolating polynomials, so it must apply to the Newton polynomials.

We now make the following general observation about polynomials: if two polynomials are equal, the coefficients of their leading terms will be equal. Because it is easy to see that the coefficient of the leading term of a Newton polynomial $p_{0,n}(x)$ is $f[x_0 , x_1 , \ldots, x_n]$, this translates into a fact about the divided differences:

$$f[x_0 , x_1 , \ldots, x_n] = \frac{f[x_0 , x_1 , \ldots, x_{n-1}] - f[x_1 , x_2 , \ldots, x_n]}{x_0 - x_n}$$

Using the original recursive formula above we have the more general version of this relation:

$$f[x_i , \dots, x_j] = \frac{f[x_i , \dots, x_{j-1}] - f[x_{i+1} , x_2 , \dots, x_j]}{x_i - x_j}$$

The general form of the Newton polynomial $p_{i,j}(x)$ also makes it easy to see that $f[x_i] = y_i$, so our recursive formula also has an appropriate set of base cases.

This recursive formula is most often used to compute the divided differences.

The last order of business here is to demonstrate that the expanded divided differences we developed above satisfy the same recursive formula. Since the bases cases satisfy $f[x_i] = y_i$ for both forms, if we can show that the expanded differences satisfy the same recursive relation as the ordinary divided differences we will have shown that both forms are the same. Examining both sides of

$$f[x_i , \dots, x_j] = \frac{f[x_i , \dots, x_{j-1}] - f[x_{i+1} , x_2 , \dots, x_j]}{x_i - x_j}$$

we can group the terms on both sides by the factors y_k they contain and then compare those terms. The coefficient of y_k on the left is

$$\frac{1}{(x_k - x_i) \cdots (x_k - x_j)}$$

while the coefficient of y_k on the right is

$$\frac{\frac{1}{(x_k - x_i) \cdots (x_k - x_{j-1})} - \frac{1}{(x_k - x_{i+1}) \cdots (x_k - x_j)}}{x_i - x_j}$$

We can combine the terms in the numerator by getting a common denominator:

$$\frac{\frac{(x_k - x_j)}{(x_k - x_i) \cdots (x_k - x_{j-1}) (x_k - x_j)} - \frac{(x_k - x_i)}{(x_k - x_i) (x_k - x_{i+1}) \cdots (x_k - x_j)}}{x_i - x_j}$$

$$= \frac{\frac{(x_k - x_j) - (x_k - x_i)}{(x_k - x_i) \cdots (x_k - x_{j-1}) (x_k - x_j)}}{x_i - x_j}$$

$$= \frac{\frac{x_i - x_j}{(x_k - x_i) \cdots (x_k - x_{j-1}) (x_k - x_j)}}{x_i - x_j}$$

$$= \frac{1}{(x_k - x_i) \cdots (x_k - x_j)}$$

This matches the term on the left, proving that the expanded divided differences satisfy the same recursive relation that the ordinary differences do. Thus, both forms of the divided differences must be equal.

A special case

A special case occurs when the points at which we seek to interpolate samples of a function $f(x)$ are equally spaced. In that case, the recursive formula for computing the divided differences can be simplified.

$$f[x_0,x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{h} = \frac{1}{h}\, \Delta f(x_0)$$

using the Aitken Δ notation from chapter 2. The second divided difference is

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0} = \frac{1}{2\,h}\left(\frac{\Delta f(x_1) - \Delta f(x_0)}{h}\right) = \frac{1}{2\,h^2}\, \Delta^2 f(x_0)$$

The k^{th} divided difference is

$$f[x_i,x_{i+1},\ldots,x_{i+k}] = \frac{1}{k!\,h^k}\, \Delta^k f(x_0)$$

A further trick we can apply is to note that when the interpolating points are equally spaced the products $(x - x_0)(x - x_1)\cdots(x - x_j)$ that appear in the Newton formula can be simplified. The trick is to compute an s so that $x = x_0 + s\,h$. Noting that $x_i = x_0 + i\,h$ gives us

$$(x - x_0)(x - x_1)\cdots(x - x_j) = ((x_0 + s\,h) - x_0)((x_0 + s\,h) - (x_0 + h))\cdots((x_0 + s\,h) - (x_0 + j\,h))$$

$$= (s\,h)((s-1)\,h)((s-2)\,h)\cdots((s-j)\,h)$$

$$= \frac{s!}{(s-j-1)!}\, h^{j+1}$$

The special form for the divided difference combined with this special form for the polynomial terms leads to a special form for the Newton polynomial:

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \frac{s(s-1)\cdots(s-k+1)}{k!}\, \Delta^k f(x_0)$$

Backward and Centered Differences

The formulas above work very nicely in situations in which we have to compute an estimate for a function $f(x)$ at a point x near the beginning of a list of data points. If x is near the end, we can replace the forward difference method shown above with an analogous *backward difference formula*.

$$P_n(x) = f(x_n) + \sum_{k=1}^{n} (-1)^k\, \frac{-s(-s-1)\cdots(-s-k+1)}{k!}\, \nabla^k f(x_n)$$

Likewise, there are also centered difference formulas. See the text for details.