Eigenvalues and Eigenvectors

Definition A vector **x** in $\mathbb{R}^n$ is an *eigenvector* of the *n* by *n* matrix *A* if there is a real number λ (called an *eigenvalue* of *A*) such that

$$A\,\mathbf{x} = \lambda\,\mathbf{x}$$

The standard technique for finding eigenvalue and eigenvector pairs is the following.

Theorem The real number λ is an eigenvalue of the matrix *A* if the system

$$(A - \lambda I)\,\mathbf{x} = 0$$

has a non-trivial solution. This system has a non-trivial solution if and only if

$$|A - \lambda I| = 0$$

The determinant in question is an n^{th} degree polynomial in λ called the *characteristic polynomial of A*.

Computing Eigenvalues and Eigenvectors

Here is a concrete example to illustrate how to find eigenvalues and eigenvectors for a matrix. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Computing the determinant of $A - \lambda I$ by Gauss elimination is messy, so this is one of the rare cases where we will be better off using the expansion by minors technique. We expand about row 1:

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 2 & 3-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (2-\lambda)\left((3-\lambda)(2-\lambda) - 2\right) - \left(2(2-\lambda)-2\right) + \left(2 - (3-\lambda)\right)$$

$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5$$

$$= -(\lambda - 5)(\lambda - 1)^2$$

This tells us that we have two eigenvalues, 5 and 1. The eigenvalue 1 is a repeated eigenvalue.

To find the associated eigenvectors we substitute the eigenvalues into the equation

$$(A - \lambda I)\,\mathbf{x} = 0$$

and solve by Gauss elimination. Since the eigenvalues are chosen to make the matrix on the left singular, we should expect rows of zeros to appear as we do the elimination.

For $\lambda = 5$ we have

$$\begin{bmatrix} -3 & 1 & 1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 1 & -3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 1 & 0 \\ 0 & -\frac{4}{3} & \frac{8}{3} & 0 \\ 0 & \frac{4}{3} & -\frac{8}{3} & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 1 & 0 \\ 0 & -\frac{4}{3} & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

When a row of zeroes appears in a Gauss elimination problem we respond by setting up a free parameter. In this case, the appropriate thing to do is to set

$$x_3 = t$$

Given that, the remaining equations tell us that

$$x_2 = 2\,t$$

$$x_1 = t$$

This tells us that the eigenvector associated with $\lambda = 5$ is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

For the eigenvalue $\lambda = 1$ we have

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

A single round of Gauss elimination yields

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In cases like this where two rows of zeros appear we proceed by introducing two free parameters:

$$x_3 = t$$

$$x_2 = s$$

$$x_1 = -s - t$$

In terms of these free parameters we can write the eigenvector

$$\mathbf{x} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

This tells us that corresponding to the eigenvalue $\lambda = 1$ we have two linearly independent eigenvectors

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Applications of eigenvalues and eigenvectors

Eigenvalues and eigenvectors will play an important role in situations in which we have an interative process that repeatedly multiplies vectors by the matrix A. One special situation occurs when $\mathbf{x}$ is an eigenvector with associated eigenvalue λ satisfying the inequality $|\lambda| < 1$. Consider what happens when we repeatedly multiply by A :

$$\mathbf{x}^{(0)} = \mathbf{x}$$

$$\mathbf{x}^{(1)} = A\,\mathbf{x} = \lambda\,\mathbf{x}$$

$$\mathbf{x}^{(2)} = A\,\mathbf{x}^{(1)} = A\,(\lambda\,\mathbf{x}) = \lambda\,(A\,\mathbf{x}) = \lambda^2\,\mathbf{x}$$

$$\vdots$$

$$\mathbf{x}^{(k)} = A\,\mathbf{x}^{(k-1)} = A\,(\lambda^{k-1}\,\mathbf{x}) = \lambda^{k-1}\,(A\,\mathbf{x}) = \lambda^k\,\mathbf{x}$$

Applying, say, the l_2 norm to the latter equation gives

$$\|\mathbf{x}^{(k)}\|_2 = |\lambda^k|\,\|\mathbf{x}\|_2$$

Given that $|\lambda| < 1$ we have that in the limit as k gets large we have

$$\lim_{k\to\infty} \|\mathbf{x}^{(k)}\|_2 = \lim_{k\to\infty} |\lambda^k|\,\|\mathbf{x}\|_2 = 0$$

and the sequence of vectors $\{\mathbf{x}^{(k)}\}$ converges to the zero vector.

This is not just a special case that applies to vectors that just happen to be eigenvectors of A. The following theorem from linear algebra gives this example broader significance.

Definition We say that the n by n matrix A has a *complete set of eigenvectors* if there are eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$ such than any vector $\mathbf{x}$ in $\mathbb{R}^n$ can be written as a linear combination of these eigenvectors:

$$\mathbf{x} = c_1\,\mathbf{x}_1 + c_2\,\mathbf{x}_2 + \cdots + c_n\,\mathbf{x}_n$$

Definition The *spectral radius* $\rho(A)$ of an n by n matrix A is the largest absolute value of an eigenvalue of A.

Theorem If A is an n by n matrix with a complete set of eigenvectors and spectral radius $\rho(A) < 1$ then for any vector $\mathbf{x}$ we have that

$$\lim_{k\to\infty} \|A^k\,\mathbf{x}\| = 0$$

Proof Let $\mathbf{x}$ be any vector. We start by writing $\mathbf{x}$ as a linear combination of eigenvectors of A:

$$\mathbf{x} = c_1\,\mathbf{x}_1 + c_2\,\mathbf{x}_2 + \cdots + c_n\,\mathbf{x}_n$$

Applying A to both sides of the equation gives

$$A\,\mathbf{x} = c_1\,A\,\mathbf{x}_1 + c_2\,A\,\mathbf{x}_2 + \cdots + c_n\,A\,\mathbf{x}_n = c_1\,\lambda_1\,\mathbf{x}_1 + c_2\,\lambda_2\,\mathbf{x}_2 + \cdots + c_n\,\lambda_n\,\mathbf{x}_n$$

where λ_1, λ_2, …, λ_n are the eigenvalues associated with the eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$. Continuing to multiply both sides by A gives

$$A^k\,\mathbf{x} = c_1\,(\lambda_1)^k\,\mathbf{x}_1 + c_2\,(\lambda_2)^k\,\mathbf{x}_2 + \cdots + c_n\,(\lambda_n)^k\,\mathbf{x}_n$$

Taking norms on both sides of the equation and using the triangle inequality for norms gives

$$\|A^k\,\mathbf{x}\| \le \left|c_1\,(\lambda_1)^k\right|\,\|\mathbf{x}_1\| + \left|c_2\,(\lambda_2)^k\right|\,\|\mathbf{x}_2\| + \cdots + \left|c_n\,(\lambda_n)^k\right|\,\|\mathbf{x}_n\|$$

By the definition of the spectral radius $\rho(A)$ we have that

$$\left|c_j\,(\lambda_j)^k\right| \le \left|c_j\,(\rho(A))^k\right|$$

Since $\rho(A) < 1$,

$$\lim_{k\to\infty}\left|c_k\,(\rho(A))^k\right| = 0$$

This is sufficient to prove that

$$\lim_{k\to\infty} \|A^k\,\mathbf{x}\| \le \lim_{k\to\infty}\left|c_1\,(\lambda_1)^k\right|\,\|\mathbf{x}_1\| + \lim_{k\to\infty}\left|c_2\,(\lambda_2)^k\right|\,\|\mathbf{x}_2\| + \cdots + \lim_{k\to\infty}\left|c_n\,(\lambda_n)^k\right|\,\|\mathbf{x}_n\| = 0$$

The theorem also motivates the following definition.

Definition A matrix A is said to be *convergent* if

$$\lim_{k \to \infty} \left(A^k\right)_{i,j} = 0$$

for all indices i and j.