Pointwise convergence of complex Fourier series

Let $f(x)$ be a periodic function with period $2\,l$ defined on the interval $[-l,l]$. The complex Fourier coefficients of $f(x)$ are

$$c_n = \frac{1}{2\,l} \int_{-l}^{l} f(s)\, e^{-i\,n\,\pi\,s/l}\, \mathrm{d}\,s$$

This leads to a Fourier series representation for $f(x)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\,n\,\pi\,x/l}$$

We have two important questions to pose here.

1. For a given x, does the infinite series converge?
2. If it converges, does it necessarily converge to $f(x)$?

We can begin to address both of these issues by introducing the partial Fourier series

$$f_N(x) = \sum_{n=-N}^{N} c_n\, e^{i\,n\,\pi\,x/l}$$

In terms of this function, our two questions become

1. For a given x, does $\lim_{N\to\infty} f_N(x)$ exist?
2. If it does, is $\lim_{N\to\infty} f_N(x) = f(x)$?

The Dirichlet kernel

To begin to address the questions we posed about $f_N(x)$ we will start by rewriting $f_N(x)$. Initially, $f_N(x)$ is defined by

$$f_N(x) = \sum_{n=-N}^{N} c_n\, e^{i\,n\,\pi\,x/l}$$

If we substitute the expression for the Fourier coefficients

$$c_n = \frac{1}{2\,l} \int_{-l}^{l} f(s)\, e^{-i\,n\,\pi\,s/l}\, \mathrm{d}\,s$$

into the expression for $f_N(x)$ we obtain

$$f_N(x) = \sum_{n=-N}^{N} \left(\frac{1}{2\,l} \int_{-l}^{l} f(s)\, e^{-i\,n\,\pi\,s/l}\, \mathrm{d}\,s \right) e^{i\,n\,\pi\,x/l}$$

$$= \int_{-l}^{l} \left(\frac{1}{2\,l} \sum_{n=-N}^{N} \left(e^{-i\,n\,\pi\,s/l}\, e^{i\,n\,\pi\,x/l} \right) \right) f(s)\ \mathrm{d}\,s$$

$$= \int_{-l}^{l} \left(\frac{1}{2\, l} \sum_{n=-N}^{N} e^{i\, n\, \pi\, (x-s)/l} \right) f(s)\ \mathrm{d}\, s$$

The expression in parentheses leads us to make the following definition. The *Dirichlet kernel* is the function defined as

$$K_N(x) = \frac{1}{2\, l} \sum_{n=-N}^{N} e^{i\, n\, \pi\, x/l}$$

In terms of the Dirichlet kernel, we can write the expression for $f_N(x)$ as

$$f_N(x) = \int_{-l}^{l} K_N(x-s) f(s)\ \mathrm{d}\, s$$

Some properties of the Dirichlet kernel

By rewriting the expression for the Dirichlet kernel, we can recogize that the Dirichlet kernel is actually a geometric series.

$$K_N(x) = \frac{1}{2\, l} \sum_{n=-N}^{N} e^{i\, n\, \pi\, x/l} = \frac{1}{2\, l} \sum_{n=-N}^{N} \left(e^{i\, \pi\, x/l} \right)^n$$

Because this is a geometric series, it can be summed explicitly.

$$K_N(x) = \frac{1}{2\, l} \sum_{n=-N}^{N} \left(e^{i\, \pi\, x/l} \right)^n = \frac{1}{2\, l} \frac{\left(e^{i\, \pi\, x/l} \right)^{N+1} - \left(e^{i\, \pi\, x/l} \right)^{-N}}{e^{i\, \pi\, x/l} - 1}$$

$$= \frac{1}{2\, l} \frac{\left(e^{i\, \pi\, x/l} \right)^{N+1/2} - \left(e^{i\, \pi\, x/l} \right)^{-(N+1/2)}}{\left(e^{i\, \pi\, x/l} \right)^{1/2} - \left(e^{i\, \pi\, x/l} \right)^{-1/2}}$$

$$= \frac{1}{2\, l} \frac{\left(e^{i\, \pi\, x/l} \right)^{N+1/2} - \left(e^{i\, \pi\, x/l} \right)^{-(N+1/2)}}{e^{i\, \pi\, x/(2 l)} - e^{-i\, \pi\, x/(2 l)}}$$

$$= \frac{\sin\left(\frac{(2 N+1)\, \pi\, x}{2\, l} \right)}{2\, l \sin\left(\frac{\pi\, x}{2\, l} \right)}$$

Some explicit integrations show that

$$\int_{-l}^{0} K_N(x)\, \mathrm{d}\, x = \int_{0}^{l} K_N(x)\, \mathrm{d}\, x = \frac{1}{2}$$

Convolutions

The integral we saw earlier

$$f_N(x) = \int_{-l}^{l} K_N(x-s) f(s) \, \mathrm{d}\, s$$

is an example of what is known as a *convolution integral*. Specifically, if $g(x)$ and $h(x)$ are two periodic functions with period $2\, l$ defined on $[-l, l]$ the *convolution of g and h* is defined by

$$(g{*}h)(x) = \int_{-l}^{l} g(x-s)\, h(s) \, \mathrm{d}\, s$$

Here is an important property of convolution integrals. From the definition we have that

$$(g{*}h)(x) = \int_{-l}^{l} g(x-s)\, h(s) \, \mathrm{d}\, s$$

If we introduce a change of variables $z = x - s$ in the integral, the integral becomes

$$\int_{x+l}^{x-l} g(z) h(x-z)(-1) \, \mathrm{d}\, z = \int_{x-l}^{x+l} g(z) h(x-z) \, \mathrm{d}\, z$$

Since both g and h are assumed to be periodic with the same period, if we shift the range of integration by a factor of x, the integral has the same value.

$$\int_{x-l}^{x+l} g(z) h(x-z) \, \mathrm{d}\, z = \int_{-l}^{l} g(z) h(x-z) \, \mathrm{d}\, z$$

Replacing the variable z with s in the final integral gives

$$(g{*}h)(x) = \int_{-l}^{l} g(x-s)\, h(s) \, \mathrm{d}\, s = \int_{-l}^{l} g(s) h(x-s) \, \mathrm{d}\, s = \left(h{*}g\right)(x)$$

This is an important symmetry property of the convolution of periodic functions.

For our present purposes, because both the Dirichlet kernel $K_N(x)$ and our function $f(x)$ are periodic, we have that

$$f_N(x) = \int_{-l}^{l} K_N(x-s)\, f(s) \, \mathrm{d}\, s = (K_N{*}f)(x) = \left(f{*}K_N\right)(x) = \int_{-l}^{l} K_N(s)\, f(x-s) \, \mathrm{d}\, s$$

This latter form is a more convenient form to work with.

The pointwise convergence theorem

A function $f(x)$ is said to be *piecewise smooth* on an interval $[-l, l]$ if the function has at most a finite number of isolated discontinuities in that interval, and at each point where the function is discontinuous it has a finite limit on either side of the discontinuity. That is,

$$\lim_{s \to x^-} f(s) = f(x-)$$

$$\lim_{s \to x^+} f(s) = f(x+)$$

both exist and are finite.

We are now in a position to state

Pointwise convergence theorem for complex Fourier series

If $f(x)$ is a piecewise smooth periodic function defined on the interval $[-l,l]$ then

$$\lim_{N\to\infty} f_N(x) = f(x)$$

whereever $f(x)$ is continuous. At points where $f(x)$ has a jump discontinuity,

$$\lim_{N\to\infty} f_N(x) = \frac{1}{2}\left(f(x\text{-}) + f(x\text{+})\right)$$

Proof We will show a somewhat stronger pair of results.

$$\lim_{N\to\infty} \int_{-l}^{0} K_N(s) f(x\text{-}s)\ \mathrm{d}\,s = \frac{1}{2}\ f(x\text{+})$$

$$\lim_{N\to\infty} \int_{0}^{l} K_N(s) f(x\text{-}s)\ \mathrm{d}\,s = \frac{1}{2}\ f(x\text{-})$$

both proofs are similar, so we will only show the proof of the second equality.

To start with, we will use a fact about the Dirichlet kernel I mentioned above.

$$\int_{0}^{l} K_N(s)\,\mathrm{d}\,s = \frac{1}{2}$$

Using this gives us

$$\frac{1}{2}\ f(x\text{-}) = \int_{0}^{l} f(x\text{-})\ K_N(s)\,\mathrm{d}\,s$$

Thus, to show that

$$\lim_{N\to\infty} \int_{0}^{l} K_N(s) f(x\text{-}s)\ \mathrm{d}\,s = \frac{1}{2}\ f(x\text{-})$$

we can instead prove the equivalent

$$\lim_{N\to\infty} \int_{0}^{l} K_N(s)\left(f(x\text{-}s) - f(x\text{-})\right)\ \mathrm{d}\,s = 0$$

Earlier I showed that

$$K_N(s) = \frac{\sin\left(\frac{(2N+1)\ \pi\, s}{2\ l}\right)}{2\ l \sin\left(\frac{\pi\, s}{2\ l}\right)}$$

Substituting this into the integral gives

$$\lim_{N\to\infty}\int_0^l \frac{\sin\left(\frac{(2N+1)\,\pi\, s}{2\,l}\right)}{2\,l\sin\left(\frac{\pi\, s}{2\,l}\right)}\big(f(x\text{-}s) - f(x\text{-})\big)\;\mathrm{d}\,s = 0$$

or

$$\lim_{N\to\infty}\int_0^l \frac{f(x\text{-}s) - f(x\text{-})}{2\,l\sin\left(\frac{\pi\, s}{2\,l}\right)}\sin\left(\frac{(2N+1)\,\pi\, s}{2\,l}\right)\;\mathrm{d}\,s = 0$$

Next, we introduce

$$F_{(x)}(s) = \frac{f(x\text{-}s) - f(x\text{-})}{2\,l\sin\left(\frac{\pi\, s}{2\,l}\right)}$$

To proceed beyond this point we are now going to need a pair of lemmas.

Lemma 1

$$\lim_{s\to 0^+} F_{(x)}(s) = \lim_{s\to 0^+}\frac{f(x\text{-}s) - f(x\text{-})}{2\,l\sin\left(\frac{\pi\, s}{2\,l}\right)} = \frac{\lim_{s\to 0^+}\left(-\,\frac{\mathrm{d}\,f}{\mathrm{d}\,x}(x\text{-}s)\right)}{\lim_{s\to 0^+}\left(\frac{\pi}{2\,l}\,\cos\left(\frac{\pi\, s}{2\,l}\right)\right)}$$

Even if x is a point of discontinuity, if we assume that f is piecewise smooth, then

$$\lim_{s\to 0^+}\left(-\,\frac{\mathrm{d}\,f}{\mathrm{d}\,x}(x\text{-}s)\right)$$

exists and is finite. Thus,

$$\lim_{s\to 0^+} F_{(x)}(s) = -\,\frac{\mathrm{d}\,f}{\mathrm{d}\,x}(x\text{-})$$

Lemma 2 (Bessel's Inequality)

If $\{\varphi_N(s)\}$ is a sequence of orthogonal functions defined on $[0,l]$ then for all N and all functions $F(s)$ we have

$$\sum_{N=0}^{\infty}\frac{\left|(F(s)\;,\;\varphi_N(s))\right|^2}{(\varphi_N(s)\;,\;\varphi_N(s))} \le (F(s),F(s))$$

Here

$$(\;,\;)$$

is any inner product for our function space. In practice, this is usually the standard complex inner product

$$(F(s)\;,\;\varphi_N(s)) = \int_0^l F(s)\;\overline{\varphi_N(s)}\;\mathrm{d}\,s$$

We now use these two lemmas to continue with the proof of our main result. We need to prove that

$$\lim_{N\to\infty} \int_0^{l} F_{(x)}(s) \sin\left(\frac{(2N+1)\,\pi\, s}{2\,l}\right) \mathrm{d}\,s = 0$$

To prove this, we apply Bessel's inequality with $F(s) = F_{(x)}(s)$ and $\varphi_N(s) = \sin\big((2N+1)\,\pi\, s/2\, l\big)$. The first thing to note here is that the sequence of functions

$$\varphi_N(s) = \sin\left(\frac{(2N+1)\,\pi\, s}{2\,l}\right)$$

is in fact a sequence of orthogonal functions defined on the interval $[0,l]$.

Now consider the inner product

$$(F(s),F(s)) = (F_{(x)}(s),F_{(x)}(s)) = \int_0^{l} \left(F_{(x)}(s)\right)^2 \mathrm{d}\,s$$

The only thing that could keep this integral from being finite is a singularity at $s = 0$. By lemma 1 above,

$$\lim_{s\to 0^+} F_{(x)}(s) = -\,\frac{\mathrm{d}\,f}{\mathrm{d}\,x}(x\text{-})$$

so there is no such singularity. Thus, the right hand side in the inequality

$$\sum_{N=0}^{\infty} \frac{(F_{(x)}(s)\ ,\ \varphi_N(s))}{(\varphi_N(s)\ ,\ \varphi_N(s))} \leq (F_{(x)}(s)\ ,\ F_{(x)}(s))$$

must be finite, and hence the sum on the left must converge.

For that sum to converge, a necessary condition is that

$$\lim_{N\to\infty} \frac{\left|(F_{(x)}(s)\ ,\ \varphi_N(s))\right|^2}{(\varphi_N(s)\ ,\ \varphi_N(s))} = 0$$

Since

$$(\varphi_N(s)\ ,\ \varphi_N(s)) = \int_0^{l} \left(\sin\left(\frac{(2N+1)\,\pi\, s}{2\,l}\right)\right)^2 \mathrm{d}\,s$$

$$= \frac{\frac{1}{2}\left(\sin\big(2\,N\,\pi\big) + 2\,N\,\pi + \pi\right) l}{2\,N\,\pi + \pi}$$

$$= \frac{l}{2}$$

saying that

$$\lim_{N\to\infty} \frac{\left|(F_{(x)}(s)\ ,\ \varphi_N(s))\right|^2}{(\varphi_N(s)\ ,\ \varphi_N(s))} = 0$$

means that we must have

$$\lim_{N\to\infty} (F_{(x)}(s)\ ,\ \varphi_N(s)) = 0$$

This translates into the condition that

$$\lim_{N\to\infty} \int_0^{l} F_{(x)}(s) \sin\left(\frac{(2N+1)\ \pi\ s}{2\ l}\right)\ \mathrm{d}\, s = 0$$

and the result is proved.