Pointwise convergence vs. uniform convergence

We saw in section 12.4 that if a function defined on $[-l,l]$ is piecewise smooth then its partial Fourier series converges to the function at every point where the function is smooth.

$$\lim_{N\to\infty} f_N(x) = f(x)$$

The result we proved in section 12.4 simply states that $f_N(x)$ will converge to $f(x)$ at each point at which $f(x)$ is smooth, but the result gives no guarantees concerning the rate of convergence. In fact, what we can expect in most applications is that at some values of x, in particular those x that are far from the discontinuities, the partial Fourier series will converge to $f(x)$ rather quickly. For x closer to a discontinuity, the partial Fourier series will still converge to $f(x)$, but will take longer to do so.

In section 12.5 we are going to investigate conditions that might be sufficient to guarantee *uniform convergence* of the partial Fourier series $f_N(x)$ to $f(x)$. A sequence of functions $f_n(x)$ converges uniformly to $f(x)$ on an interval $[-l,l]$ if for any $\epsilon > 0$ there is an N such that

$$|f(x) - f_n(x)| < \epsilon$$

for all x in $[-l,l]$ for all $n \geq N$.

One thing we can say already is that if $f(x)$ is piecewise smooth, the sequence of partial Fourier sums $f_n(x)$ will *not* converge uniformly to $f(x)$. As we will see below, placing further restrictions on the smoothness of $f(x)$ will be sufficient to guarantee uniform convergence.

Smoothness and decay properties of Fourier coefficients

The accompanying Mathematica notebook demonstrates an interesting relationship between the smoothness of a function and the rate of decay of its complex Fourier coefficients c_n.

- A piecewise continuous function has Fourier coefficients that decay as $1/n$.
- A continuous function with discontinuous first derivative has Fourier coefficients that decay as $1/n^2$.
- A continuous function with continuous first derivative but discontinuous second derivative has Fourier coefficients that decay as $1/n^3$.

There appears to be a general relationship here. A continuous periodic function whose first k derivatives are all continuous but whose k+1 derivative is discontinuous should have Fourier coefficients that decay at a rate of $1/n^{k+2}$.

We can actually prove this general relationship by using an argument based on integration by parts. Suppose that $f(x)$ is a continuous periodic function on $[-l,l]$ whose first k derivatives are all continuous (including the condition that the function and all its first k derivatives are continuous across the boundaries of the interval).

The Fourier coefficients are computed by the usual formula.

$$c_n = \frac{1}{2\,l} \int_{-l}^{l} f(x)\; e^{-i\,\pi\,n\,x/l}\,d\,x$$

Integration by parts gives us

$$c_n = \frac{1}{2\,l} \int_{-l}^{l} f(x)\; e^{-i\,\pi\,n\,x/l}\,d\,x$$

$$= \left(\frac{1}{2\,l}\;\frac{1}{-i\,\pi\;n/l}\;f(x)\;e^{-i\,\pi\,n\,x/l}\right)\Big|_{-l}^{\,l} - \frac{1}{2\,l}\;\frac{1}{-i\,\pi\;n/l}\;\int_{-l}^{l} f\,'(x)\,e^{-i\,\pi\,n\,x/l}\;d\,x$$

$$= i\left(\frac{\cos(\pi\;n)\;f(l)}{2\;\pi\;n} - \frac{\cos(\pi\;n)\;f(-l)}{2\;\pi\;n}\right) - \frac{1}{2\;l}\;\frac{1}{-i\;\pi\;n/l}\;\int_{-l}^{l} f\,'(x)\,e^{-i\,\pi\,n\,x/l}\;d\,x$$

$$= -\frac{i}{2\;\pi\;n}\int_{-l}^{l} f\,'(x)\,e^{-i\,\pi\,n\,x/l}\;d\,x$$

We can continue playing this integration by parts game until we reach a point at which either the function or one of its derivatives is discontinuous at either the boundaries or an interior point. Each round of integration by parts that the function will support contributes another factor of $1/n$, so that a function with k continuous derivatives can survive this process up to the $1/n^{k+2}$ term. If the function has a derivative that is discontinuous at either an interior point or at the boundaries, the boundary terms generated in the integration by parts will cease to cancel when we reach the round involving that derivative, giving us a bound on the power of n that appears in the Fourier coefficient.

Decay of Fourier coefficients and uniform convergence

We have just seen that there is a connection between the degree of smoothness of a function and the rate at which its Fourier coefficients decay. We will not establish a connection between that rate of decay and the convergence properties of the sequence of partial Fourier sums.

To demonstrate that a sequence of partial Fourier sums $f_n(x)$ converges uniformly to $f(x)$ on an interval $[-l,l]$ we need to place a uniform bound on

$$\left|f(x) - f_n(x)\right| = \left|f(x) - \sum_{k=-n}^{n} c_k\;e^{i\,k\,\pi\,x/l}\right| = \left|\sum_{k=-\infty}^{-n-1} c_k\;e^{i\,k\,\pi\,x/l} + \sum_{k=n}^{\infty} c_k\;e^{i\,k\,\pi\,x/l}\right|$$

The latter terms are the so called *remainder terms* in the full Fourier expansion for $f(x)$. Our job is to place a bound on those remainders.

First, note that we can write

$$\left|\sum_{k=-\infty}^{-n-1} c_k\;e^{i\,k\,\pi\,x/l} + \sum_{k=n}^{\infty} c_k\;e^{i\,k\,\pi\,x/l}\right| \le \sum_{k=-\infty}^{-n-1}\left|c_k\;e^{i\,k\,\pi\,x/l}\right| + \sum_{k=n}^{\infty}\left|c_k\;e^{i\,k\,\pi\,x/l}\right|$$

$$\leq \sum_{k=-\infty}^{-n-1} |c_k| + \sum_{k=n}^{\infty} |c_k|$$

We now see that the smoothness of $f(x)$ will play a direct role in this estimate, because if the smoothness is sufficient to guarantee that

$$|c_k| \leq \frac{M}{k^r}$$

then we will see that

$$|f(x) - f_n(x)| \leq \sum_{k=-\infty}^{-n-1} |c_k| + \sum_{k=n}^{\infty} |c_k| \leq \sum_{k=-\infty}^{-n-1} \frac{M}{k^r} + \sum_{k=n}^{\infty} \frac{M}{k^r}$$

If $r \geq 2$ the series

$$\sum_{k=-\infty}^{\infty} \frac{M}{k^r}$$

is a convergent series, so the remainder terms

$$\sum_{k=-\infty}^{-n-1} \frac{M}{k^r} + \sum_{k=n}^{\infty} \frac{M}{k^r}$$

can be made arbitrarily small for n large enough. Note that this bound is independent of x and is hence a uniform bound.

Putting this together with the results we developed earlier concerning the decay of Fourier coefficients now gives us a theorem.

Theorem If $f(x)$ is a continuous, periodic function on $[-l,l]$ with a piecewise continuous first derivative, then the complex Fourier coefficients of $f(x)$ satisfy the inequality

$$|c_k| \leq \frac{M}{k^2}$$

for k large enough and the sequence of partial Fourier sums

$$f_n(x) = \sum_{k=-n}^{n} c_k \, e^{i k \pi x/l}$$

converges uniformly to $f(x)$.