Solving the Heat Equation by the Finite Element Method

Consider the heat equation with Dirichlet boundary conditions

$$\rho\, c\, \frac{\partial u}{\partial t} - \kappa\, \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

$$u(0,t) = u(l,t) = 0$$

$$u(x,0) = \psi(x)$$

To apply the Galerkin method to this equation we start by multiplying both sides of the PDE by test functions $v(x)$ from $C_D^{\,2}[0,l]$ and integrating to make a weak form of the PDE.

$$\int_0^l \left(\rho\, c\, \frac{\partial u}{\partial t} - \kappa\, \frac{\partial^2 u}{\partial x^2}\right) v(x)\, \mathrm{d}\,x = \int_0^l f(x,t)\; v(x)\, \mathrm{d}\,x$$

As usual, we apply integration by parts one time to convert the form of the second term on the left:

$$\int_0^l \left(- \kappa\, \frac{\partial^2 u}{\partial x^2}\right) v(x)\, \mathrm{d}\,x = - \kappa\, \left(\frac{\partial u}{\partial x}\; v(x)\right)\Big|_0^{\,l} + \kappa \int_0^l \frac{\partial u}{\partial x}\, \frac{\partial v}{\partial x}\, \mathrm{d}\,x$$

Since $v(x)$ vanishes at the boundary we have

$$\int_0^l \rho\, c\, \frac{\partial u}{\partial t}\; v(x) + \kappa\, \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}\, \mathrm{d}\,x = \int_0^l f(x,t)\; v(x)\, \mathrm{d}\,x$$

This is the weak form of the heat equation. As before, we will select N functions $\varphi_i(x)$ that form a basis for a subspace V_N of $C_D^{\,2}[0,l]$. This time around we have to assume that the approximate solution $u_N(x,t)$ takes the form

$$u_N(x,t) = \sum_{j=1}^{N} \alpha_j(t)\; \varphi_j(x)$$

Since the solution depends on both x and t we have to assume that the coefficients of this combination are functions of t.

Substituting this approximate solution with test function $v(x) = \varphi_i(x)$ into the weak form gives

$$\int_0^l \rho\, c\, \sum_{j=1}^{N} \alpha_j^{'}(t)\; \varphi_j(x)\; \varphi_i(x) + \kappa\, \sum_{j=1}^{N} \alpha_j(t)\; \frac{\mathrm{d}\varphi_j(x)}{\mathrm{d}x} \frac{\mathrm{d}\varphi_i(x)}{\mathrm{d}x}\, \mathrm{d}\,x = \int_0^l f(x,t)\; \varphi_i(x)\, \mathrm{d}\,x$$

If we introduce *mass matrix M* whose i,j entry is

$$M_{i,j} = \int_0^l \rho\, c\; \varphi_j(x)\; \varphi_i(x)\, \mathrm{d}\,x$$

a stiffness matrix K whose i,j entry is

$$K_{i,j} = \int_0^l \kappa\, \frac{\mathrm{d}\varphi_j(x)}{\mathrm{d}x} \frac{\mathrm{d}\varphi_i(x)}{\mathrm{d}x}\, \mathrm{d}\,x$$

a vector $\mathbf{f}(t)$ whose j entry is

$$\mathbf{f}_j(t) = \int_0^l f(x,t)\, \varphi_j(x)\, \mathrm{d}\, x$$

and a vector $\alpha(t)$ whose j entry is $\alpha_j(t)$ we can write the equation above as a system of ODEs for the vector $\alpha(t)$ of unknown coefficients $\alpha_j(t)$:

$$M\, \alpha'(t) = -K\, \alpha(t) + \mathbf{f}(t)$$

This system of equations has an initial condition given by the requirement that

$$u_N(x,0) = \sum_{j=1}^{N} \alpha_j(0)\, \varphi_j(x) \approx \sum_{j=1}^{N} \psi(x_j)\, \varphi_j(x)$$

The accompanying Mathematica notebook will demonstate several different methods that can be used to solve this system of ODEs.