Green's Function for the Wave Equation - D'Alembert's method

We have already seen that the homogeneous wave equation on the real line

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$u(x,0) = \psi(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \gamma(x)$$

can be solved by D'Alembert's method:

$$u(x,t) = \frac{\psi(x-ct)}{2} + \frac{\psi(x+ct)}{2} + \frac{1}{2\,c} \int_{x-ct}^{x+ct} \gamma(s)\,\mathrm{d}\,s$$

The Green's function method is aimed at solving the non-homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

$$u(x,0) = 0$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

In order to solve this problem, we have to convert the non-homogeneous problem to a homogeneous problem. Once again, the technique that allows us to accomplish this is a variant of Duhamel's principle. We start by solving

$$\frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial x^2} = 0$$

$$v(x,0) = 0$$

$$\frac{\partial v}{\partial t}(x,0) = f(x,s)$$

where s is a parameter. This can be solved directly by d'Alembert's method, giving

$$v(x,t;s) = \frac{1}{2\,c} \int_{x-ct}^{x+ct} f(y,s)\,\mathrm{d}\,y$$

Duhamel's principle then gives us a solution

$$u(x,t) = \int_0^t v(x,t-s;s)\,\mathrm{d}\,s = \frac{1}{2\,c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s$$

Two adjustments are needed before we can put this solution into the standard Green's function form

$$u(x,t) = \int_0^\infty \int_{-\infty}^\infty G(x,t;y,s) f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s$$

The first is that we need to insert a function of y with the characteristic that it is 1 when

$$x - c\,(t-s) \le y \le x + c\,(t-s)$$

and 0 when y is outside this range. Rewriting the inequality slightly gives

$$-c\,(t-s) \le y - x \le c\,(t-s)$$

or

$$|y-x| \le c(t-s)$$

or

$$0 \le c(t-s) - |y-x|$$

The function that does what we want here is the Heaviside function:

$$H(z) = \begin{cases} 1 & z \ge 0 \\ 0 & z < 0 \end{cases}$$

In terms of this Heaviside function we have that

$$\int_{x-c(t-s)}^{x+c(t-s)} \frac{1}{2\,c} f(y,s)\,\mathrm{d}\,y = \int_{-\infty}^{\infty} \frac{1}{2\,c} H(c(t-s) - |y-x|)\,f(y,s)\,\mathrm{d}\,y$$

As for the outer integral, all we have to note is that when $s > t$ we have that both $c(t-s)$ and $-|y-x|$ are negative and the term $H(c(t-s) - |y-x|)$ will automatically be 0. This allows us to extend the outer integral from t all the way to ∞ for free, since the integrand will be 0 over that entire additional range.

Thus

$$u(x,t) = \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\,c} H(c(t-s) - |y-x|)\,f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s$$

and the Green's function for the wave equation on the real line is

$$G(x,t;y,s) = \frac{1}{2\,c} H(c(t-s) - |y-x|)$$

Green's function on a bounded interval - Fourier Series method

We have already seen that the Fourier series solution to the PDE

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

$$u(0,t) = u(l,t) = 0$$

$$u(x,0) = 0$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

with Dirichlet boundary conditions can be obtained by writing the solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi}{l} x\right)$$

In terms of the Fourier coefficients of the forcing function

$$c_n(t) = \frac{2}{l}\int_0^l f(x,t) \sin\left(\frac{n\pi}{l} x\right) \mathrm{d}\,x$$

we get a set of ODEs to solve for the coefficients $a_n(t)$:

$$\frac{\mathrm{d}^2 a_n(t)}{\mathrm{d}t^2} + \frac{c^2 n^2 \pi^2}{l^2} a_n(t) = c_n(t)$$

$$a_n(0) = 0$$

$$a_n'(0) = 0$$

Once again, we can apply Duhamel's principle. We start by solving

$$\frac{\mathrm{d}^2 v(t)}{\mathrm{d}t^2} + \frac{c^2 n^2 \pi^2}{l^2} v(t) = 0$$

$$v(0) = 0$$

$$v'(0) = c_n(s)$$

which has solution

$$v(t;s) = \frac{l}{c\, n\, \pi}\, c_n(s)\sin\left(\frac{c\, n\, \pi}{l}\, t\right)$$

This leads to a solution for $a_n(t)$:

$$a_n(t) = \int_0^t v(t\text{-}s;s)\, \mathrm{d}\,s = \int_0^t \frac{l}{c\, n\, \pi}\, c_n(s)\sin\left(\frac{c\, n\, \pi}{l}\, (t\text{-}s)\right) \mathrm{d}\,s$$

Next, we substitute

$$c_n(s) = \frac{2}{l}\int_0^l f(y,s) \sin\left(\frac{n\pi}{l} y\right) \mathrm{d}\,y$$

to get

$$a_n(t) = \int_0^t \frac{l}{c\, n\, \pi} \frac{2}{l}\int_0^l f(y,s) \sin\left(\frac{n\pi}{l} y\right) \mathrm{d}\,y\, \sin\left(\frac{c\, n\, \pi}{l}\, (t\text{-}s)\right) \mathrm{d}\,s$$

Rearranging slightly gives

$$a_n(t) = \int_0^t \int_0^l \left(\frac{2}{c\,n\,\pi} \sin\left(\frac{n\,\pi}{l}\,y\right) \sin\left(\frac{c\,n\,\pi}{l}\,(t\text{-}s)\right) \right) f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s$$

Substituting this expression into

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\sin\left(\frac{n\,\pi}{l}\,x\right)$$

gives

$$u(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t \int_0^l \left(\frac{2}{c\,n\,\pi} \sin\left(\frac{n\,\pi}{l}\,y\right) \sin\left(\frac{c\,n\,\pi}{l}\,(t\text{-}s)\right) \right) f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s \right) \sin\left(\frac{n\,\pi}{l}\,x\right)$$

and finally

$$u(x,t) = \int_0^t \int_0^l \sum_{n=1}^{\infty} \left(\frac{2}{c\,n\,\pi} \sin\left(\frac{c\,n\,\pi}{l}\,(t\text{-}s)\right) \sin\left(\frac{n\,\pi}{l}\,x\right) \sin\left(\frac{n\,\pi}{l}\,y\right) \right) f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s$$

The Green's function for the wave equation with Dirichlet boundary conditions on $[0,l]$ is

$$G(x,t;y,s) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{c\,n\,\pi} \sin\left(\frac{c\,n\,\pi}{l}\,(t\text{-}s)\right) \sin\left(\frac{n\,\pi}{l}\,x\right) \sin\left(\frac{n\,\pi}{l}\,y\right) \right) & s \le t \\ 0 & s > t \end{cases}$$

Alternative derivation

The Green's function we just derived is quite a bit less elegant than the Green's function we derived earlier for the wave equation on the real line. A natural question to ask is whether or not we can trick that earlier solution into solving the wave equation with Dirichlet boundary conditions on $[0,l]$.

One way to accomplish this is to modify the forcing function. Since we seek a solution that vanishes at 0 and l for all time, one way we may be able to accomplish this is by introducing a forcing function that has special symmetry properties designed to force the solution to vanish at these points. For example, if I want to construct a solution that vanishes at the origin for all time, I could introduce a forcing function with the following symmetry:

$$f(-x,t) = -f(x,t) \text{ for } 0 \le x \le l$$

This symmetry will guarantee that any wave travelling toward the boundary at $x = 0$ from the right will be met by an equal and opposite wave travelling toward $x = 0$ from the left. Similarly, we can get waves to cancel at the right boundary by forcing

$$f(l+x,t) = -f(x,t) \text{ for } 0 \le x \le l$$

More generally, what we need to do is to introduce a function $\tilde{f}(x,t)$ that is an *odd periodic extension* of the original forcing function $f(x,t)$ defined on $0 \le x \le l$.

$$\tilde{f}(x,t) = \begin{cases} f(x - n\,l,t) & 0 \le x - n\,l \le l \text{ and } n \text{ is even} \\ -f(x - n\,l,t) & 0 \le x - n\,l \le l \text{ and } n \text{ is odd} \end{cases}$$

In terms of this periodic extension, we have that

$$u(x,t) = \int_0^\infty \int_{-\infty}^\infty \frac{1}{2\,c} H(c(t-s) - |y-x|)\,\tilde{f}(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

for all x in $[0,l]$.

The big problem that remains with this solution is that the inner integral is over the wrong range. For a problem on $[0,l]$ we want that integral to be an integral from $y = 0$ to $y = l$.

The key to getting this to happen is to note that the odd periodic extension can be written as the sum of an infinite number of translates of itself.

$$\tilde{f}(x,t) = \sum_{n=-\infty}^{\infty} f_n(x,t)$$

where

$$f_n(x,t) = \begin{cases} \tilde{f}(x,t) & (2\,n - 1)\,l \le x \le (2\,n + 1)\,l \\ 0 & \mathit{otherwise} \end{cases}$$

This can also be written

$$f_n(x,t) = \tilde{f}(x,t)\,H(l - |x - 2\,n\,l|)$$

This allows us to write

$$u(x,t) = \int_0^\infty \int_{-\infty}^\infty \frac{1}{2\,c} H(c(t-s) - |y-x|) \sum_{n=-\infty}^{\infty} \tilde{f}(y,s)\,H(l - |y - 2\,n\,l|)\,\mathrm{d}y\,\mathrm{d}s$$

$$= \sum_{n=-\infty}^{\infty} \int_0^\infty \int_{-\infty}^\infty \frac{1}{2\,c} H(c(t-s) - |y-x|)\tilde{f}(y,s)\,H(l - |y - 2\,n\,l|)\,\mathrm{d}y\,\mathrm{d}s$$

$$= \sum_{n=-\infty}^{\infty} \int_0^\infty \int_{(2n-1)l}^{(2n+1)l} \frac{1}{2\,c} H(c(t-s) - |y-x|)\tilde{f}(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

Since the inner integral is periodic in y with period $2\,l$, we can shift the y integration back to the interval $[-l,l]$ giving us

$$u(x,t) = \sum_{n=-\infty}^{\infty} \int_0^\infty \int_{-l}^{l} \frac{1}{2\,c} H(c(t-s) - |x - y - 2\,n\,l|)\tilde{f}(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

Finally, we use the fact that $\tilde{f}(y,s)$ is an odd function in the interval $[-l,l]$ to write the first half of the inner integral

$$\int_{-l}^{0} \frac{1}{2\,c} H(c(t-s) - |x - y - 2\,n\,l|)\tilde{f}(y,s)\,\mathrm{d}y = -\int_0^l \frac{1}{2\,c} H(c(t-s) - |x + y - 2\,n\,l|) f(y,s)\,\mathrm{d}y$$

Thus

$$\int_{-l}^{l} \frac{1}{2c} H(c(t\text{-}s) - |x - y - 2nl|) \tilde{f}(y,s)\, \mathrm{d}y =$$
$$\int_{0}^{l} \frac{1}{2c} \big(H(c(t\text{-}s) - |x - y - 2nl|) - H(c(t\text{-}s) - |x + y - 2nl|)\big) f(y,s)\, \mathrm{d}y$$

We read off from this that

$$G(x,t;y,s) = \sum_{n=-\infty}^{\infty} \frac{1}{2c} \big(H(c(t\text{-}s) - |x - y - 2nl|) - H(c(t\text{-}s) - |x + y - 2nl|)\big)$$