Differential Operators

Consider the differential operator

$$L(u(x)) = -T \frac{d^2}{dx^2}(u(x))$$

acting on the space $C^2[0,l]$ of twice-continuously differentiable functions on the closed interval $[0,l]$. This operator is a linear operator, because

$$L(\alpha\, u(x) + \beta\, v(x)) = -T \frac{d^2}{dx^2}(\alpha\, u(x) + \beta\, v(x)) = \alpha \left(-T \frac{d^2}{dx^2}(u(x))\right) + \beta\left(-T \frac{d^2}{dx^2}(v(x))\right)$$

We want to bring the methods of chapter 3 to bear on this operator to solve equations of the form

$$L(u(x)) = f(x)$$

Boundary Conditions

One problem with the operator as described above is that it does not have a trivial null space. The null space of this operator is the set of all functions that satisfy

$$L(u(x)) = -T \frac{d^2}{dx^2}(u(x)) = 0$$

It is easy to see that any function of the form

$$u(x) = a\, x + b$$

satisfies this equation, giving the operator L a non-trivial null space. This in turn makes the solutions to the differential equation non-unique.

The usual fix for this problem is to impose extra conditions on the equation. If we require that solutions to

$$-T \frac{d^2}{dx^2}(u(x)) = 0$$

also satisfy the *Dirichlet boundary conditions*

$$u(0) = u(l) = 0$$

then the only function in the null space will be the function

$$u(x) = 0$$

Another way to look at this is to say that we have restricted the original operator to act on a subspace $C_D^2[0,1]$ of $C^2[0,l]$, called the *Dirichlet subspace*. This subspace consists of all twice continuously differentiable functions on [0,l] that vanish at the boundary.

Symmetry

The space of functions that we are operating on, $C_D^2[0,1]$, is also an inner product space. On this space we use the inner product

$$(u,v) = \int_0^l u(x)v(x)\,dx$$

The operator L is a symmetric operator on this space:

$$(L\,u\,,\,v\,) = \int_0^l \left(-\,T\,\frac{d^2}{dx^2}(u(x))\right)v(x)\,dx = v(x)\,\left(-T\,\frac{d}{dx}u(x)\right)\Big|_0^{\,l} + \int_0^l \left(T\,\frac{d}{dx}u(x)\right)\left(\frac{d}{dx}v(x)\right)dx$$

$$= 0 + \,T\,u(x)\,\left(\frac{d}{dx}v(x)\right)\Big|_0^{\,l} - \int_0^l T\,u(x)\left(\frac{d^2}{dx^2}(v(x))\right)dx$$

$$= \int_0^l u(x)\left(-\,T\,\frac{d^2}{dx^2}(v(x))\right)dx$$

$$= (\,u\,,\,L\,v\,)$$

Here we have used integration by parts twice and twice applied the fact that both $u(x)$ and $v(x)$ vanish at both 0 and l.

Eigenvalues and Eigenfunctions

An eigenfunction of the differential operator L is a function on $C_D^2[0,1]$ that satisfies the equation

$$L\,u(x) = -\,T\,\frac{d^2}{dx^2}(u(x)) = \lambda\,u(x)$$

or equivalently

$$u''(x) + \frac{\lambda}{T}\,u(x) = 0$$

with boundary conditions

$$u(0) = u(l) = 0$$

Using methods from Math 210, we can solve this equation and see that solutions take the form

$$u(x) = \sin\left(\sqrt{\frac{\lambda}{T}}\,x\right)$$

where λ has to be chosen so that

$$\sqrt{\frac{\lambda}{T}} = \frac{n\,\pi}{l}$$

for $n = 1, 2, 3, \ldots$ so that $u(x)$ will vanish at $x = l$. Thus we see that the operator has an infinite number of eigenvalues

$$\lambda_n = T\,\frac{n^2\,\pi^2}{l^2}$$

with associated eigenfunctions

$$u_n(x) = \sin\left(\frac{n\,\pi}{l}\,x\right)$$

Solving by the Spectral Method

We have seen that if a linear operator has a complete set of eigenfunctions and eigenvalues we can use the spectral method to solve problems of the form

$$L(u(x)) = f(x)$$

by using eigenfunction expansions. We seek to write the solution

$$u(x) = \sum_{n=1}^{\infty} c_n\,u_n(x)$$

We solve for the unknown coefficients by writing the right hand side as an expansion in the eigenfunctions:

$$f(x) = \sum_{n=1}^{\infty} d_n\,u_n(x)$$

Once we have determined the expansion coefficients d_n we can solve the problem.

$$L(u(x)) = L\left(\sum_{n=1}^{\infty} c_n\,u_n(x)\right) = \sum_{n=1}^{\infty} c_n\,\lambda_n\,u_n(x) = f(x) = \sum_{n=1}^{\infty} d_n\,u_n(x)$$

Since the eigenfunctions form a basis for our space and are independent, this equation can be solved by setting

$$c_n\,\lambda_n = d_n$$

or

$$c_n = \frac{d_n}{\lambda_n}$$

The only thing left to do here is to compute the d_n coefficients. We do this by using the inner product:

$$(f(x)\ ,\ u_k(x)) = \left(\sum_{n=1}^{\infty} d_n\ u_n(x)\ ,\ u_k(x)\right) = \sum_{n=1}^{\infty} (d_n\ u_n(x)\ ,\ u_k(x)) = d_k\ (u_k(x)\ ,\ u_k(x))$$

or

$$d_k = \frac{(f(x)\ ,\ u_k(x))}{(u_k(x)\ ,\ u_k(x))}$$

These coefficients, called *Fourier coefficients*, are computed by using the integral definition of the inner product.

$$d_k = \frac{\int_0^l f(x)\ u_k(x)\ dx}{\int_0^1 u_k(x)\ u_k(x)\ dx} = \frac{\int_0^l f(x)\ \sin\left(\frac{k\ \pi}{l}\ x\right)\ dx}{\int_0^l \sin^2\left(\frac{k\ \pi}{l}\ x\right)\ dx}$$

Noting that

$$\int_0^l \sin^2\left(\frac{k\ \pi}{l}\ x\right)\ dx = \frac{1}{4}\ \left(-\frac{\sin(2\ \pi\ k)}{\pi\ k} + 2\right)\ l = \frac{l}{2}$$

this simplifies to

$$d_k = \frac{2}{l}\ \int_0^l f(x)\ \sin\left(\frac{k\ \pi}{l}\ x\right)\ dx$$

Computing a finite approximation

The expansion of the function $f(x)$ in terms of eigenfunctions

$$f(x) = \sum_{n=1}^{\infty} d_n\ u_n(x)$$

has an infinite number of terms. In practice, we can compute only finitely many Fourier coefficients. This produces a finite approximation

$$f(x) \approx f_N(x) = \sum_{n=1}^{N} d_n\ u_n(x)$$

which in turn leads to a finite approximation for the solution:

$$L(u_N(x)) = f_N(x)$$

where

$$u_N(x) = \sum_{n=1}^{N} c_n\ u_n(x)$$

where as before

$$c_n = \frac{d_n}{\lambda_n}$$

$$d_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n \pi}{l} x\right) dx$$

We leave it as an open question for now whether the function $u_N(x)$ is the closest approximation to the actual solution to the equation

$$L(u(x)) = f_N(x)$$

in the subspace of functions that take the form

$$u_N(x) = \sum_{n=1}^{N} c_n \, u_n(x)$$