The method of characteristics

The method of characteristics is a method that is frequently used to solve first order PDEs. The simplest such PDEs take the form

$$a \frac{\partial u(x,y)}{\partial x} + b \frac{\partial u(x,y)}{\partial y} = 0$$

$$u(x,0) = u_0(x)$$

The method uses a change of variables to simplify the problem. Specifically, we seek new variables s and t such that after the change of variables the PDE simplifies to

$$\frac{\partial u(x(s,t),y(s,t))}{\partial t} = 0$$

We can use the chain rule to help us determine what the change of variables should be by noting that

$$0 = \frac{\partial u(x(s,t),y(s,t))}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$$

Comparing this with the original PDE

$$a \frac{\partial u(x,y)}{\partial x} + b \frac{\partial u(x,y)}{\partial y} = 0$$

shows that we should set

$$\frac{\partial x}{\partial t} = a$$

$$\frac{\partial y}{\partial t} = b$$

A further requirement on the new variables is that the *initial curve*, which is the curve on which the initial conditions are specified, should correspond to the curve $t = 0$.

In the simplest first order PDEs the initial curve is the line $y = 0$. This suggests that we should choose

$$y(s,0) = 0$$

$$x(s,0) = s$$

Putting all of these requirements together gives us a pair of simple PDEs that the functions $x(s,t)$ and $y(s,t)$ must satisfy

$$\frac{\partial x(s,t)}{\partial t} = a$$

$$x(s,0) = s$$

$$\frac{\partial y(s,t)}{\partial t} = b$$

$$y(s,0) = 0$$

These equations are easily solved to yield

$$x(s,t) = a\ t + s$$

$$y(s,t) = b\ t$$

Once we have solved for x and y we can finish solving the PDE. If we introduce

$$v(s,t) = u(x(s,t),y(s,t))$$

the PDE reduces to

$$\frac{\partial v(s,t)}{\partial t} = 0$$

$$v(s,0) = u(x(s,0),y(s,0)) = u_0(s)$$

which has solution

$$v(s,t) = u_0(s)$$

The final step is to invert the equations for x and y to express s and t as functions of x and y:

$$t = y/b$$

$$s = x - a\ t = x - \frac{a}{b}\ y$$

This allows us to express the solution $v(s,t)$ as a function of x and y:

$$v(s,t) = u_0(s) = u_0(x - \frac{a}{b}\ y) = u(x,y)$$

An outline of the method

To summarize, we seek a change of variables that maps the initial curve to the line $t = 0$ and simplifies the original PDE to

$$\frac{\partial v(s,t)}{\partial t} = 0$$

This condition basically says that along curves of fixed s the solution is a constant. These curves of fixed s are called *characteristic curves*. Solving the PDEs for x and y as functions of s and t and inverting will allow us to compute the characteristic curves.

Here is a complete outline of the method applied to a more general first order linear PDE of the form

$$a(x,y)\ \frac{\partial u}{\partial x} + b(x,y)\ \frac{\partial u}{\partial y} + c(x,y)\ u = d(x,y)$$

$$u(x,0) = u_0(x)$$

1. Require that

$$\frac{\partial x(s,t)}{\partial t} = a(x(s,t),y(s,t))$$

$$x(s,0) = s$$

$$\frac{\partial y(s,t)}{\partial t} = b(x(s,t),y(s,t))$$

$$y(s,0) = 0$$

2. Solve these PDEs for $x(s,t)$ and $y(s,t)$.

3. Invert the formulas to obtain $s(x,y)$ and $t(x,y)$.

4. Solve the PDE

$$\frac{\partial v(s,t)}{\partial t} + c(x(s,t),y(s,t))\ v(s,t) = d(x(s,t),y(s,t))$$

$$v(s,0) = u_0(x(s,0))$$

5. Construct the solution

$$u(x,y) = v(s(x,y),t(x,y))$$

An example

This is example is based on example 8.4 from the text.

$$y\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + x\ u = x$$

$$u(x,0) = x$$

We now follow the steps outlined above.

1. Solve the equations for x and y as functions of s and t. (Typically we solve the easier of the two PDEs first and use that result to help solve the other PDE.)

$$\frac{\partial y(s,t)}{\partial t} = 1$$

$$y(s,0) = 0$$

$$y(s,t) = t$$

$$\frac{\partial x(s,t)}{\partial t} = y(s,t) = t$$

$$x(s,0) = s$$

$$x(s,t) = s + \frac{t^2}{2}$$

2. Invert these equations to obtain

$$t = y$$

$$s = x - \frac{t^2}{2} = x - \frac{y^2}{2}$$

3. The PDE for v becomes

$$\frac{\partial v(s,t)}{\partial t} + c(x(s,t),y(s,t))\ v(s,t) = d(x(s,t),y(s,t))$$

$$\frac{\partial v(s,t)}{\partial t} + x(s,t)\ v(s,t) = x(s,t)$$

$$\frac{\partial v(s,t)}{\partial t} + (s + \frac{t^2}{2})\ v(s,t) = s + \frac{t^2}{2}$$

$$v(s,0) = x(s,0) = s$$

We can solve the latter PDE by treating s as a parameter and constructing an ODE for $v(s,t) = w_s(t)$:

$$w_s{}'(t) + (s + \frac{t^2}{2})\ w_s(t) = s + \frac{t^2}{2}$$

$$w_s(t) = s$$

This ODE has solution

$$w_s(t) = 1 + (s\text{-}1)\ e^{-s\,t - t^2/2}$$

4. We substitute the expressions for s and t to obtain

$$u(x,y) = 1 - (1 - x + \frac{y^2}{2})\ e^{-x\,y + y^3/3}$$

Quasi-linear first order PDEs

The method described above can be extended to deal with first order *quasi-linear* PDEs

$$a(x,y,u)\ \frac{\partial u}{\partial x} + b(x,y,u)\ \frac{\partial u}{\partial y} = c(x,y,u)$$

$$u(x,1) = u_0(x)$$

The major difference here is that instead of solving PDEs for x and y and then solving a PDE for v we will have to treat all three PDEs as a single, coupled system:

$$\frac{\partial x}{\partial t} = a(x,y,v)\ ;\ x(s,0) = s$$

$$\frac{\partial y}{\partial t} = b(x,y,v)\ ;\ y(s,0) = 1$$

$$\frac{\partial v}{\partial t} = c(x,y,v)\ ;\ v(s,0) = u_0(s)$$

Here is example 8.7 in the text.

$$u\ \frac{\partial u}{\partial x} + y\ \frac{\partial u}{\partial y} = x$$

$$u(x,1) = 2\ x$$

The system to solve is

$$\frac{\partial x}{\partial t} = v\ ;\ x(s,0) = s$$

$$\frac{\partial y}{\partial t} = y\ ;\ y(s,0) = 1$$

$$\frac{\partial v}{\partial t} = x\ ;\ v(s,0) = 2\ s$$

Treating s as a parameter and solving this as a system of linear ODEs gives

$$x(s,t) = \frac{3\ e^{t} - e^{-t}}{2}\ s$$

$$y(s,t) = e^{t}$$

$$u(s,t) = \frac{3\ e^{t} + e^{-t}}{2}\ s$$

Inverting these equations gives

$$t = \ln\ y$$

$$s = \frac{2\ x}{3\ y - 1/y}$$

$$u(x,y) = \frac{3\ y + 1/y}{2}\ \frac{2\ x}{3\ y - 1/y} = \frac{(3\ y^{2} + 1)\ x}{3\ y^{2} - 1}$$

Limitations of the method

The examples above demonstrate that this method works for a variety of first order PDEs. There are however a couple of 'choke points' in the method where a particular problem may prove difficult or impossible to solve.

The first of these comes when we have to solve the equations involving $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial t}$, and possibly also $\frac{\partial v}{\partial t}$. Without too much effort we can concoct examples in which the resulting sets of equations are simply too hard to solve. Consider this minor variation on the last example.

$$\frac{\partial x}{\partial t} = v\ ;\ x(s,0) = s$$

$$\frac{\partial y}{\partial t} = y\ ;\ y(s,0) = 0$$

$$\frac{\partial v}{\partial t} = x\ v\ ;\ v(s,0) = 2\ s$$

This is a system of nonlinear ODEs, and can not be solved by any standard solution method.

The second problem comes when we manage to solve for $v(s,t)$ and then discover that the equations for $x(s,t)$ and $y(s,t)$ are just too difficult to invert. This makes it practically impossible to solve for $u(x,y)$.

A third problem is that this method relies on the initial curve not also being a characteristic curve. If that happens, we may find that the original PDE has no solutions, or an infinite number of solutions.