Polynomial Interpolation

The polynomial interpolation problem is the problem of constructing a polynomial that passes through or *interpolates* $n+1$ data points (x_0, y_0), (x_1, y_1), ... , (x_n, y_n). In chapter 3 we are going to see several techniques for constructing interpolating polynomials.

Lagrange Interpolation

To construct a polynomial of degree n passing through $n+1$ data points (x_0, y_0), (x_1, y_1), ... , (x_n, y_n) we start by constructing a set of *basis polynomials* $L_{n,k}(x)$ with the property that

$$L_{n,k}(x_j) = \begin{cases} 1 \text{ when } j = k \\ 0 \text{ when } j \neq k \end{cases}$$

These basis polynomials are easy to construct. For example for a sequence of x values $\{x_0, x_1, x_2, x_3\}$ we would have the four basis polynomials

$$L_{3,0}(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$L_{3,1}(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$L_{3,2}(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$

$$L_{3,3}(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

Once we have constructed these basis functions, we can form the n^{th} degree *Lagrange interpolating polynomial*

$$L(x) = \sum_{k=0}^{n} y_k \, L_{n,k}(x)$$

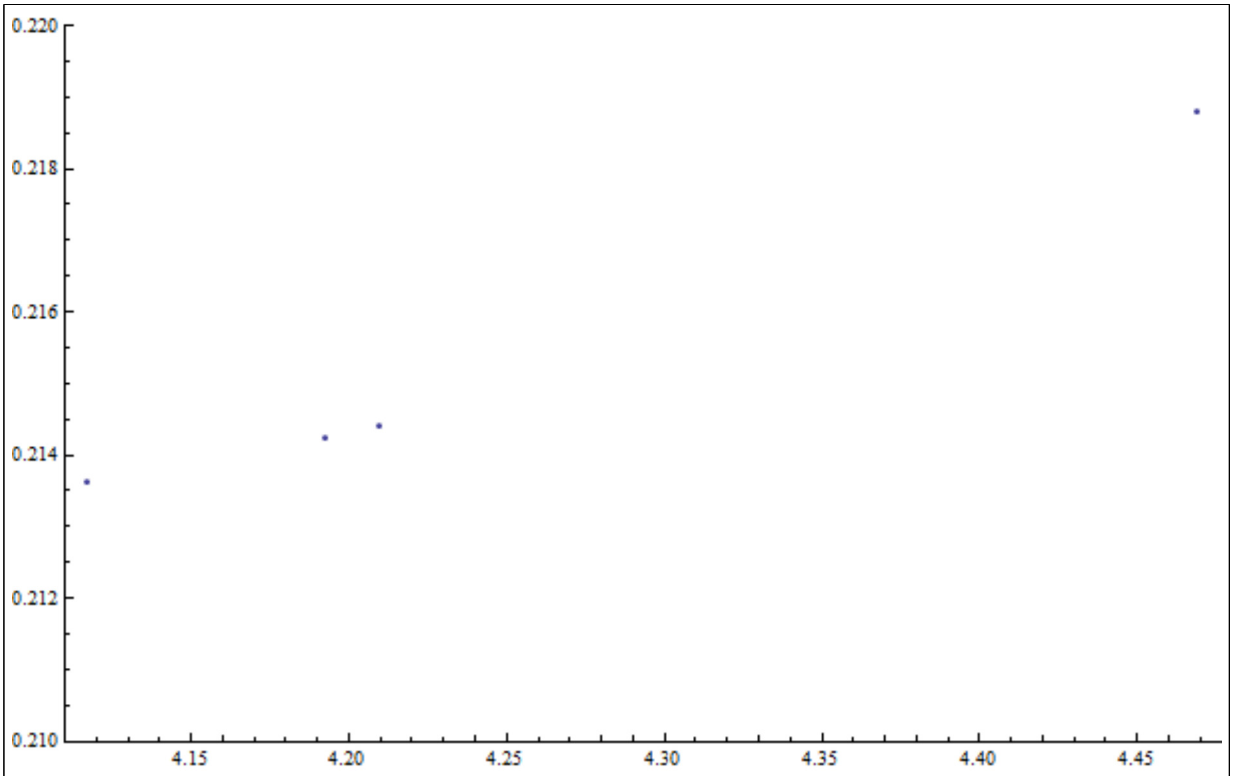
This polynomial does what we want it to do, because when $x = x_j$ every one of the basis functions vanishes, except for $L_{n,j}(x)$, which has value 1. Thus $L(x_j) = y_j$ for every j and the polynomial interpolates each one of the data points in the original data set.

An example

Here is a set of data points.

x	y
4.1168	0.213631
4.19236	0.214232
4.20967	0.21441
4.46908	0.218788

Here is a plot of these points showing that they line up along a curve, but the curve is not quite linear.



-data-

To construct the Lagrange interpolating polynomial of degree 3 passing through these points we first compute basis functions:

$$L_{3,0}(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$= \frac{(x - 4.19236)(x - 4.20967)(x - 4.46908)}{(4.1168 - 4.19236)(4.1168 - 4.20967)(4.1168 - 4.46908)}$$

$$L_{3,1}(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$= \frac{(x - 4.1168)(x - 4.20967)(x - 4.46908)}{(4.19236 - 4.1168)(4.19236 - 4.20967)(4.19236 - 4.46908)}$$

$$L_{3,2}(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$

$$= \frac{(x - 4.1168)(x - 4.19236)(x - 4.46908)}{(4.20967 - 4.1168)(4.20967 - 4.19236)(4.20967 - 4.46908)}$$

$$L_{3,3}(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

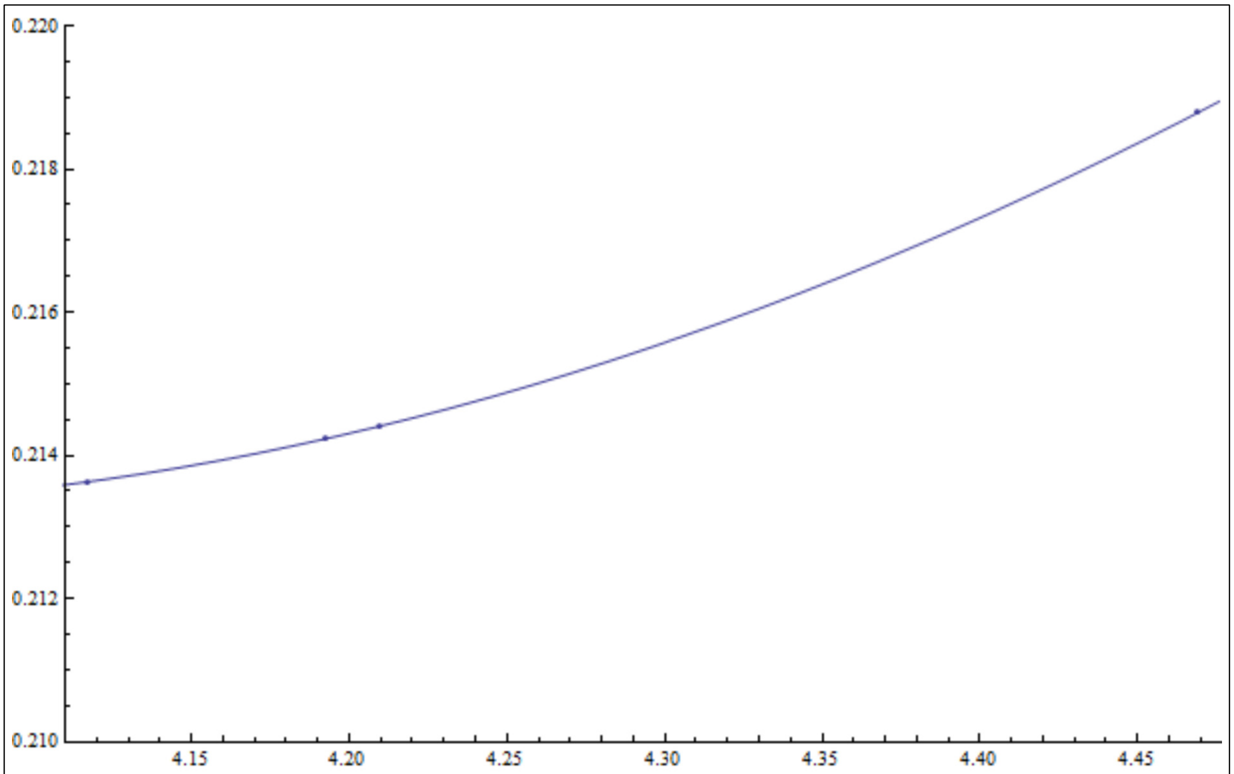
$$= \frac{(x - 4.1168)(x - 4.19236)(x - 4.20967)}{(4.46908 - 4.1168)(4.46908 - 4.19236)(4.46908 - 4.20967)}$$

From these we construct the interpolating polynomial:

$$L(x) = y_0\, L_{3,0}(x) + y_1\, L_{3,1}(x) + y_2\, L_{3,2}(x) + y_3\, L_{3,3}(x)$$

$$= -0.00355245\, x^3 + 0.0695519\, x^2 - 0.386008\, x + 0.871839$$

Here are the original data points plotted along with the interpolating polynomial.



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$y = -0.00355245\, x^3 + 0.0695519\, x^2 - 0.386008\, x + 0.871839$

Error Estimate

For each polynomial interpolation method we examine in chapter 3 we will want to also generate an estimate of how accurate the method the method is for a particular application. For example, suppose that the data points (x_0, y_0), (x_1, y_1), ... , (x_n, y_n) we interpolate by this method are actually generated by an underlying function $f(x)$. That is, $y_k = f(x_k)$ for all k. How large can we then expect the error

$$|f(x) - L(x)|$$

to be over a range of x including the x values we interpolated?

The following theorem answers this question.

Theorem If the function $f(x)$ has $n+1$ continuous derivatives on some interval $[a,b]$ and the polynomial $P(x)$ is the Lagrange interpolating polynomial constructed to interpolate a set of points $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$ with $x_k \in [a,b]$ for all k then for each x in $[a,b]$ there is a $\xi(x)$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

Proof If x is any one of the points x_k we have that

$$f(x_k) = P(x_k) + \frac{f^{(n+1)}(\xi(x_k))}{(n+1)!}(x_k - x_0)(x_k - x_1)\cdots(x_k - x_k)\cdots(x_k - x_n)$$

or

$$f(x_k) = f(x_k) + 0$$

For $x \neq x_k$ for any k, we define a function

$$g(t) = f(t) - P(t) - [f(x) - P(x)]\,\frac{(t - x_0)(t - x_1)\ldots(t - x_n)}{(x - x_0)(x - x_1)\ldots(x - x_n)}$$

Note that $g(t)$ vanishes as each of the $n+1$ points $t = x_k$. By construction, $g(t)$ also vanishes at $t = x$. This means that $g(t)$ vanishes at a total of $n+2$ distinct points. Note also that $g(t)$ has $n+1$ continuous derivatives on the interval $[a,b]$. By the generalized version of Rolle's theorem, there must be a $\xi(x)$ in $[a,b]$ such that

$$g^{(n+1)}(\xi(x)) = 0$$

or

$$\begin{aligned} 0 &= g^{(n+1)}(\xi(x)) \\ &= f^{(n+1)}(\xi(x)) - P^{(n+1)}(\xi(x)) \\ &\quad - [f(x) - P(x)]\,\frac{d^{n+1}}{dt^{n+1}}\left(\frac{(t - x_0)(t - x_1)\ldots(t - x_n)}{(x - x_0)(x - x_1)\ldots(x - x_n)}\right)\Bigg|_{t = \xi(x)} \end{aligned}$$

or

$$0 = f^{(n+1)}(\xi(x)) - [f(x) - P(x)]\,\frac{(n+1)!}{(x - x_0)(x - x_1)\ldots(x - x_n)}$$

Solving this equation for $f(x)$ gives

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$