Notes on

Complex Impedance and Phasors

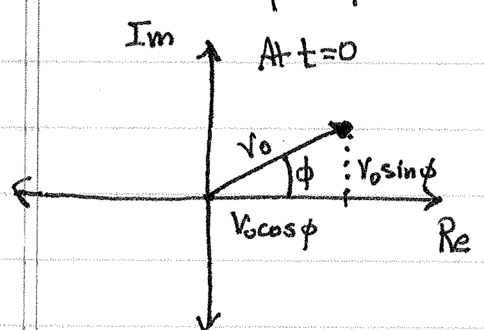
- Analysis of A.C. circuits is aided by the use complex exponential functions and "phasors" to represent oscillating circuit quantities.

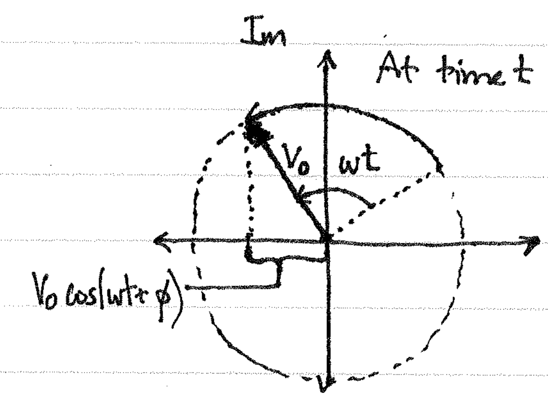
- Recall Euler's Formula: $e^{j\theta} = \cos\theta + j\sin\theta$ $\qquad j = \sqrt{-1}$

- Represent oscillating quantities (like voltage): $\tilde{\mathbb{V}}(t) = V_0 e^{j\omega t} = V_0\cos\omega t + jV_0\sin\omega t$
  the real part of this function is the value that would be measured at time t.

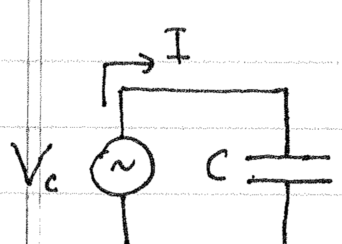
- A phase shift can be incorporated into the (complex) amplitude:
  $$\tilde{\mathbb{V}}(t) = V_0 e^{j(\omega t+\phi)} = V_0\cos(\omega t+\phi) + jV_0\sin(\omega t+\phi)$$
  $$= V_0 e^{j\phi} e^{j\omega t}$$
  or $\tilde{\mathbb{V}}(t) = \tilde{V}_0 e^{j\omega t}$ $\qquad$ where $\tilde{V}_0 = V_0 e^{j\phi} = V_0\cos\phi + jV_0\sin\phi$

- Complex oscillating quantities can be visualized as vectors rotating in the complex plane.





- Relation between voltage (across) and current (through) a capacitor:



Kirchoff loop rule: $V_c - \dfrac{Q}{C} = 0$

Take derivative: $\dfrac{dV_c}{dt} - \dfrac{1}{C}\dfrac{dQ}{dt} = 0$

or ... $\dfrac{dV_c}{dt} = \dfrac{I}{C}$

Replace V_c with $\tilde{\mathbb{V}}_c$ and I with $\tilde{\mathbb{I}}$ ... complex exponential functions.

Then.. $\dfrac{d\tilde{\mathbb{V}}_c}{dt} = j\omega\tilde{\mathbb{V}}_c$ from derivative of exponential function.

Substitute in... $j\omega\tilde{\mathbb{V}}_c = \frac{1}{C}\tilde{\mathbb{I}}$ or... $\tilde{\mathbb{V}}_c = \frac{1}{j\omega C}\tilde{\mathbb{I}} = \underbrace{\frac{-j}{\omega C}}\tilde{\mathbb{I}}$

- Impedance of a capacitor : $\boxed{\tilde{Z}_c = \dfrac{-j}{\omega C}}$

  analogous to resistance since it relates current to voltage.

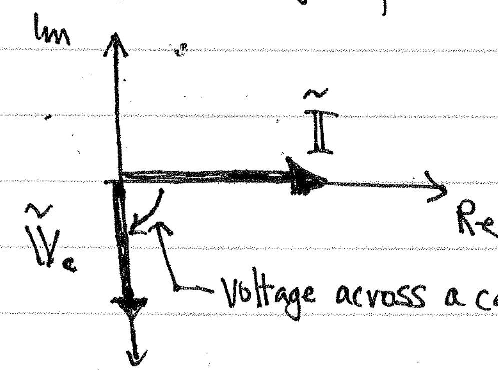
  But... is large at low frequency and small at high frequency.

- In general, a linear circuit element (or the effective combination of elements) has a complex impedance, so that current and voltage are related by...

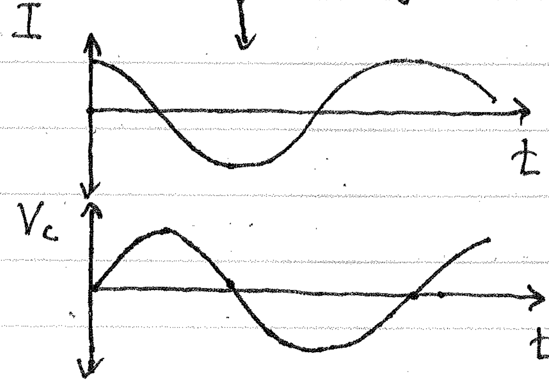
$$\boxed{\tilde{\mathbb{V}} = \tilde{\mathbb{I}}\tilde{Z}}$$

complex but constant in time ... generally depends on frequency however. Units are ohms (Ω).

- Phasors for capacitor : choose to draw phasors at the instant when the phasor for the current $\tilde{\mathbb{I}}$ is on the positive real axis. Then $\tilde{\mathbb{V}}_c$ lies on negative imaginary axis... $\tilde{\mathbb{V}}_c = \underbrace{\frac{-j}{\omega C}}_{\text{pure imaginary impedance}}\underset{\text{real}}{\tilde{\mathbb{I}}}$



Voltage across a capacitor lags the current by $90° = \frac{\pi}{2}$ radians

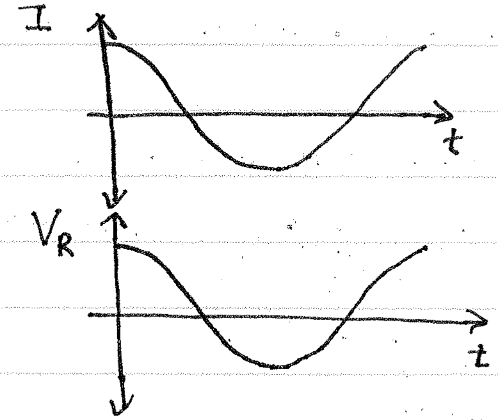


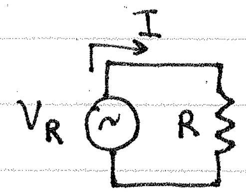
- Impedance of a resistor: $V_R = IR$ Ohm's Law

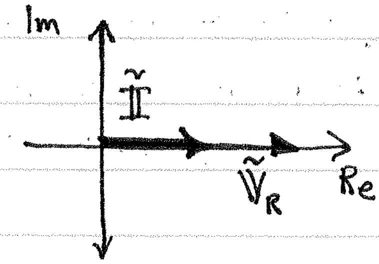
or $\rightarrow$ $\tilde{\mathbb{V}}_R = \tilde{\mathbb{I}} R$

$\boxed{\tilde{Z}_R = R}$ Impedance of a resistor is real ... and equal to the resistance.

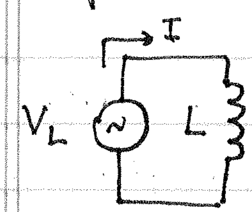
Voltage and current are in-phase with each other.







- Impedance of an inductor:



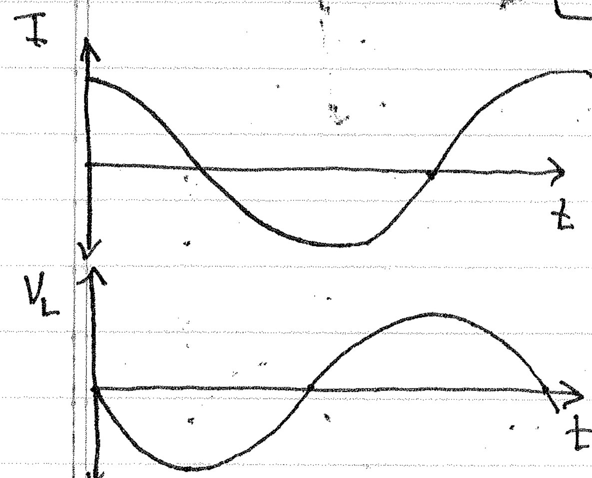
Kirchoff loop rule: $V_L - L\frac{dI}{dt} = 0$

convert to complex notation: $\tilde{\mathbb{V}}_L = L\frac{d\tilde{\mathbb{I}}}{dt}$

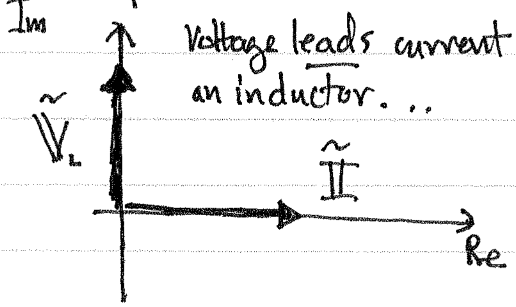
Since $\tilde{\mathbb{I}} = \tilde{I}_0 e^{j\omega t}$ $\quad \frac{d\tilde{\mathbb{I}}}{dt} = j\omega \tilde{I}_0 e^{j\omega t} = j\omega \tilde{\mathbb{I}}$

Therefore, ... $\tilde{\mathbb{V}}_L = j\omega L \tilde{\mathbb{I}}$

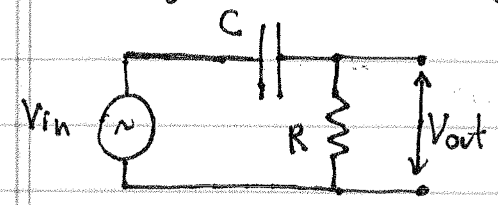
$\boxed{\tilde{Z}_L = j\omega L}$ Impedance of an inductor is small at low frequency and large at high frequency ... opposite of a capacitor.



Voltage leads current in an inductor...



- Combining element... RC High Pass Filter



Kirchoff loop rule using complex notation:

$$\tilde{V}_{in} - \tilde{I}Z_c - \tilde{I}\tilde{Z}_R = 0$$

or $$\tilde{V}_{in} = \tilde{I}\underbrace{\left(\tilde{Z}_R + \tilde{Z}_c\right)}$$

Impedance of the series combination of resistor and capacitor...

$$\tilde{Z}_{eff} = \tilde{Z}_R + \tilde{Z}_c = R - \frac{j}{\omega C}$$

$$\tilde{I} = \frac{\tilde{V}_{in}}{Z_{eff}} = \frac{\tilde{V}_{in}}{R - j/\omega C}$$

$$\tilde{V}_{out} = \tilde{V}_R = \tilde{I}R = \underbrace{\frac{R}{R - j/\omega C}}\tilde{V}_{in}$$

Note similarity to voltage divider relation!

$$V_{out_0} e^{j\phi} e^{j\omega t} = \frac{R}{R - j/\omega C} V_{in_0} e^{j\omega t}$$

cancel all time dependence

$$V_{out_0} e^{j\phi} = \left(\frac{R}{R - j/\omega C}\right) V_{in_0}$$

phase shift between input and output voltages.

$$\frac{V_{out_0}}{V_{in_0}} e^{j\phi} = \frac{R}{R - j/\omega C} \cdot \frac{(R + j/\omega C)}{(R + j/\omega C)} = \frac{R^2 + jR/\omega C}{R^2 + (1/\omega C)^2} \quad \begin{matrix} \div R^2 \\ \div R^2 \end{matrix}$$

$$\frac{V_{out_0}}{V_{in_0}} e^{j\phi} = \frac{1 + j/\omega RC}{1 + \left(\frac{1}{\omega RC}\right)^2}$$

To get $\frac{V_{out_0}}{V_{in_0}}$, take absolute value of complex quantity on right side...

$$\frac{V_{out_0}}{V_{in_0}} = \frac{\sqrt{1+\left(\frac{1}{\omega RC}\right)^2}}{\left(1+\left(\frac{1}{\omega RC}\right)^2\right)}$$ ← $\sqrt{\ }$ of sum of Re part squared plus Im part squared.

OR...
$$\boxed{\frac{V_{out_0}}{V_{in_0}} = \frac{1}{\sqrt{1+\left(\frac{1}{\omega RC}\right)^2}}} \quad \text{OR} \quad \frac{1}{\sqrt{1+\left(f_0/f\right)^2}}$$

where $f_0 = \frac{1}{2\pi RC}$

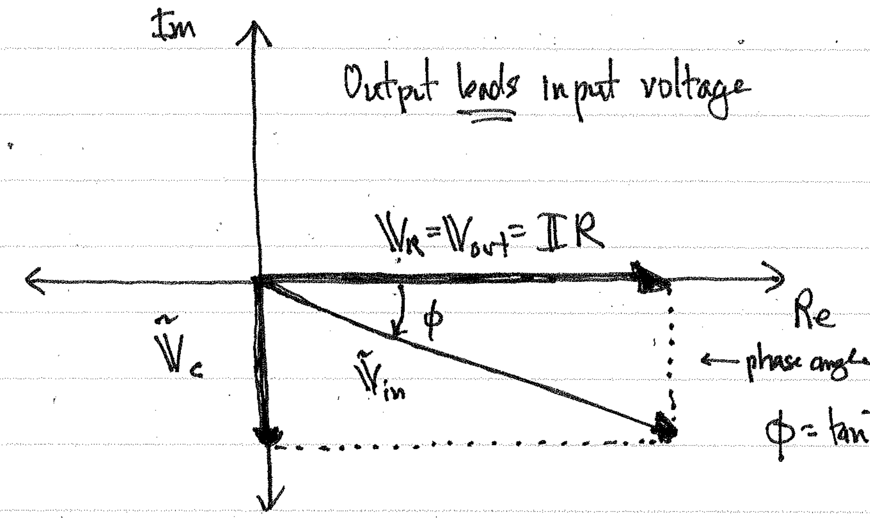
and $f = \frac{\omega}{2\pi}$

To get phase shift ... $\phi = \tan^{-1}\left(\frac{\mathrm{Im}(\ \ )}{\mathrm{Re}(\ \ )}\right)$

↑ right side of expression on previous page.

$$\boxed{\phi = \tan^{-1}\left(\frac{1}{\omega RC}\right)} \quad \text{or} \quad \phi = \tan^{-1}\left(\frac{f_0}{f}\right)$$

- Phasor diagram for RC HighPass Circuit:



$$\phi = \tan^{-1}\left(\frac{-V_c}{V_R}\right) = \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

Kirchoff's loop rule for complex phasors. $\tilde{V}_{in} - \tilde{V}_c - \tilde{V}_R = 0$

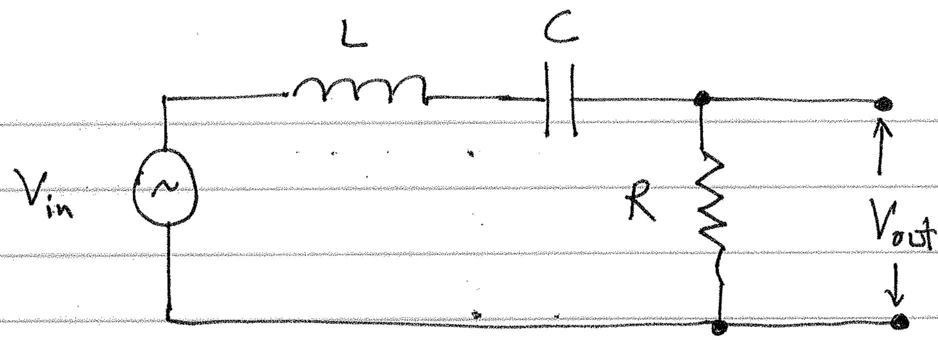
or... $\tilde{V}_{in} = \tilde{V}_c + \tilde{V}_R$ Pythagorus $V_{in_0}^2 = V_{c_0}^2 + V_{R_0}^2$

↑ also V_{out_0}

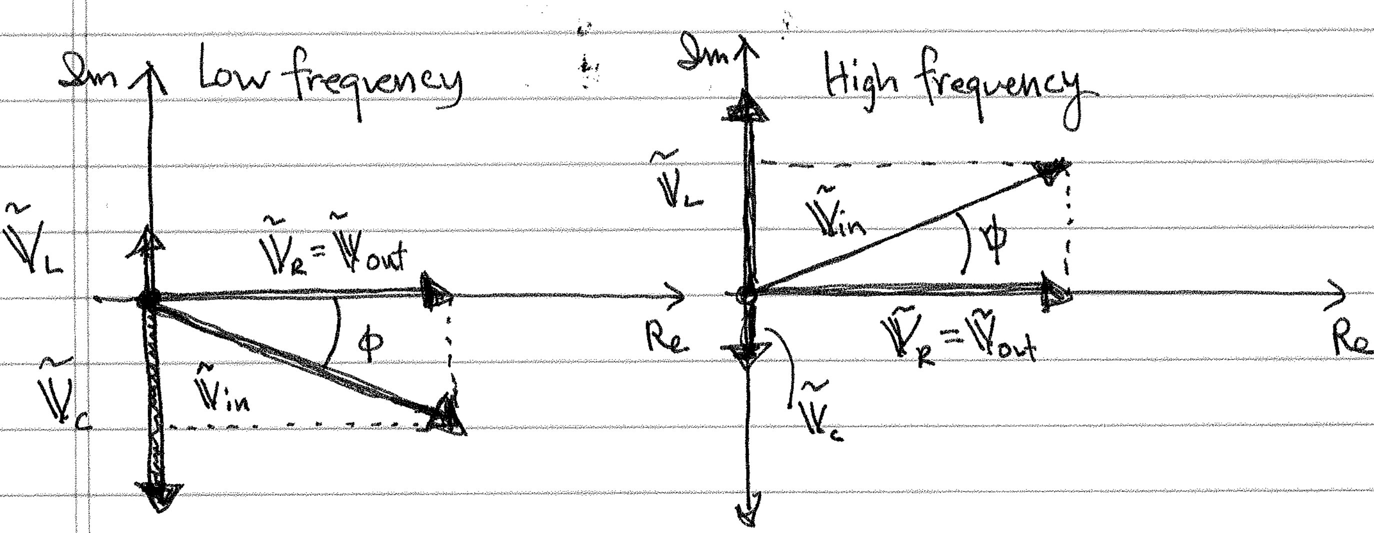
$$V_{in_0} = \sqrt{I_0^2\left(\frac{1}{\omega C}\right)^2 + I_0^2 R^2} = I_0 \underbrace{\sqrt{\left(\frac{1}{\omega C}\right)^2 + R^2}}_{\text{magnitude of complex impedance.}}$$

$$V_{out_0} = I_0 R = \frac{R}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} V_{in_0} \overset{\text{OR}}{\Rightarrow} \frac{V_{out_0}}{V_{in_0}} = \frac{1}{\sqrt{1+\left(\frac{1}{\omega RC}\right)^2}}$$

- RLC Series Resonant Circuit :



Phasor diagrams



Kirchoff : $\tilde{V}_{in} - \underbrace{\left(\tilde{V}_L + \tilde{V}_C + \tilde{V}_{R/out}\right)}_{\text{Use Pythagoream theorem to relate amplitudes of these quantities}} = 0$

$$V_{in} = \sqrt{\left(\omega L I_0 - \frac{1}{\omega C} I_0\right)^2 + R^2 I_0^2} = I_0 \sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}$$

$$V_{out} = V_R = I_0 R = \frac{R}{\sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}} \cdot V_{in}$$

$$\frac{V_{out}}{V_{in}} = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{1}{\sqrt{1 + \frac{\omega^2 L^2}{R^2} \underbrace{\left(1 - \frac{1}{\omega^2 LC}\right)^2}_{\text{Resonance when this goes to zero}}}}$$

• Resonant frequency $\omega_0 = \frac{1}{\sqrt{LC}}$ $f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}$

$$\frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{1+\underbrace{\left(\frac{\omega_0 L}{R}\right)}^2\left(\frac{\omega}{\omega_0}\right)^2\left(1-\frac{\omega_0^2}{\omega^2}\right)^2}}$$

define the "quality factor" $Q_0 \equiv \frac{\omega_0 L}{R}$

→ sharpness of the resonance peak

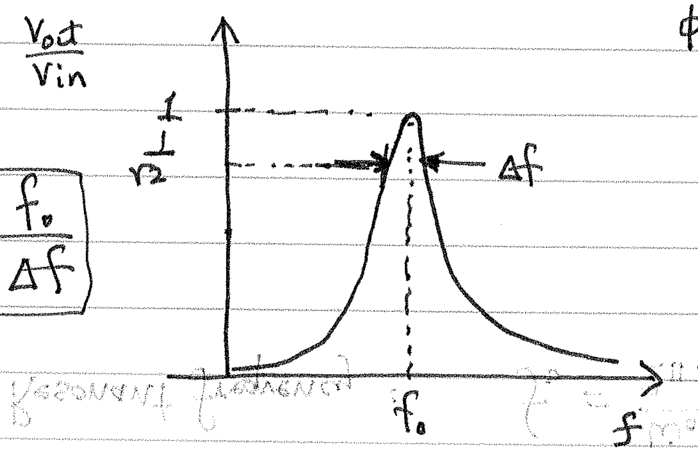
$$\boxed{\frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{1+Q_0^2\left(\frac{\omega}{\omega_0}\right)^2\left(1-\left(\frac{\omega_0}{\omega}\right)^2\right)^2}}}$$

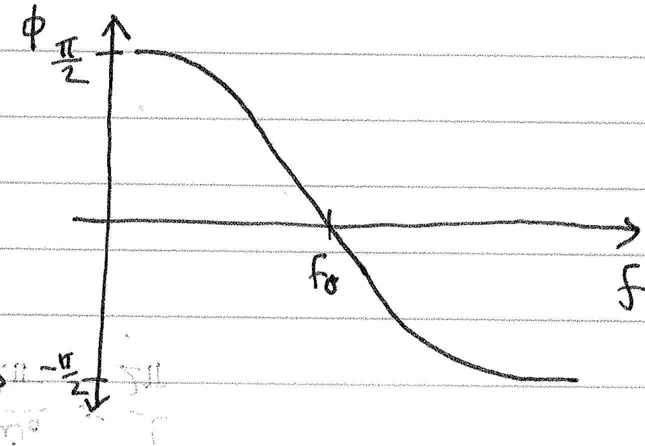
• Phase shift (see phasor diagram):

$$\phi = \tan^{-1}\left(\frac{I_0\left(\frac{1}{\omega C}-\omega L\right)}{I_0 R}\right) = \boxed{\tan^{-1}\left[\frac{1}{\omega RC}-\frac{\omega L}{R}\right]}$$

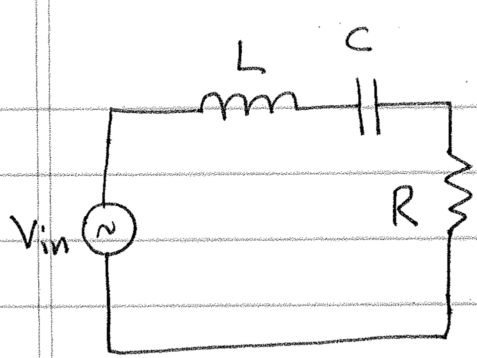
can also be written as... $\boxed{\phi = \tan^{-1}\left[Q_0\left(\frac{\omega}{\omega_0}\right)\left(\left(\frac{\omega_0}{\omega}\right)^2-1\right)\right]}$

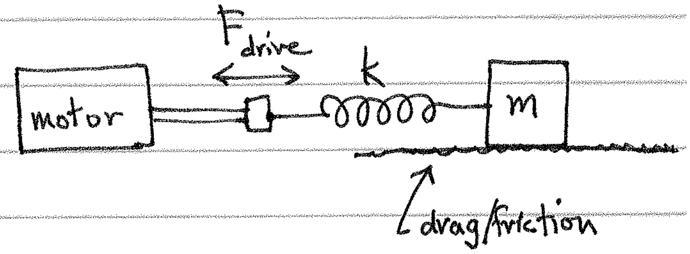
$$\boxed{Q_0 = \frac{f_0}{\Delta f}}$$





- Mechanical Analog of LRC Resonant Circuit





Newton's 2^{nd} Law

$$V_{in} - L\frac{dI}{dt} - \frac{Q}{C} - IR = 0 \qquad F_{drive} - kx - Dv = ma$$

drag force

D = drag coefficient

or.. $V_{in} - \frac{1}{C}Q - IR = L\frac{dI}{dt}$

$$Q \longleftrightarrow x$$

$$I = \frac{dQ}{dt} \longleftrightarrow v = \frac{dx}{dt}$$

$$\frac{dI}{dt} \longleftrightarrow a = \frac{dv}{dt}$$

$$L \longleftrightarrow m$$

$$\frac{1}{C} \longleftrightarrow k$$

$$V_{in} \longleftrightarrow F_{drive}$$

$$R \longleftrightarrow D$$

Resonant freq. $\omega_0 = \frac{1}{\sqrt{LC}} \longleftrightarrow \omega_0 = \sqrt{\frac{k}{m}}$