PHYSICS 160
Spring 2009

SOLUTIONS to Problem Set #1

2-10 Deriving the Lorentz transformation equations from Einstein's postulates.

- First assume the following form for the <u>linear</u> transformation equations for the x-coordinate

(1) $x' = \gamma(x - vt)$ inverse txn: (2) $x = \gamma(x' + vt')$

where γ is not yet specified, but <u>is</u> independent of x, t, x', t'.

That γ is the same for equations (1) and (2) follows from postulate I.

- Now consider a light pulse launched from $x = x' = 0$ at $t = t' = 0$ when the origins of K and K' coincide.

According to the second postulate, the position of the light pulse along the x-axis and x'-axis is...

(3) $x = ct$ and (4) $x' = ct'$

~~Substituting (3) in (1) and (4) into (2)....~~

- Substituting (3) and (4) into (1) and (2) and determine γ:

(5) $ct' = \gamma(ct - vt)$ and (6) $ct = \gamma(ct' + vt')$

or... $ct' = \gamma t(c - v)$ and $ct = \gamma t'(c + v)$

or... $t' = \gamma t(1 - \frac{v}{c})$ and $t = \gamma t'(1 + \frac{v}{c})$

substitute

$$t = \gamma^2 t \underbrace{(1 - \tfrac{v}{c})(1 + \tfrac{v}{c})}_{(1 - (\frac{v}{c})^2)}$$

cancel

So... $\gamma^2 = \dfrac{1}{1 - (\frac{v}{c})^2}$ or $\boxed{\gamma = \dfrac{1}{\sqrt{1 - (v/c)^2}}}$

2-10 – continued

- Substitute (1) into (2) to get a transformation equation for t'.

$$x = \gamma\left[\gamma(x-vt) + vt'\right]$$ Solve for t'

$$x = \gamma^2 x - \gamma^2 vt + \gamma v t'$$

$$t' = \frac{1}{\gamma v}\left[x(1-\gamma^2) + \gamma^2 vt\right]$$

$$t' = \gamma\left[t - \frac{(\gamma^2-1)}{\gamma^2 v}x\right]$$

$$\left[\frac{\gamma^2-1}{\gamma^2 v}\right] = \frac{1}{v}\left[1 - \frac{1}{\gamma^2}\right]$$

$$= \frac{1}{v}\left[1-\left(1-\left(\frac{v}{c}\right)^2\right)\right]$$

$$= +\frac{v}{c^2}$$

So... $\boxed{t' = \gamma\left[t - \frac{vx}{c^2}\right]}$

And this completes the derivation of the Lorentz transformation equations.

$\boxed{2\text{-}13}$ Two events occur in an inertial frame K at...

1. $x_1 = a$, $t_1 = 2a/c$, $y_1 = 0$, $z_1 = 0$
2. $x_2 = 2a$ $t_2 = 3a/2c$, $y_2 = 0$, $z_2 = 0$

In what frame K' will these events appear to occur at the same time? Describe the motion of system K'.

Start with Lorentz transformation equation for time...

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

In terms of intervals, or differences between spacetime coordinates for two events...

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)$$

Two find the frame in which the two events occure simultaneously, set $\Delta t' = 0$

$$0 = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)$$ and solve for v....

cancel

$$\frac{v\Delta x}{c^2} = \Delta t \quad \text{or} \quad \boxed{v = \frac{c^2 \Delta t}{\Delta x}}$$

$$\Delta t = \frac{3a}{2c} - \frac{2a}{c} = -\frac{a}{2c}$$

$$\Delta x = 2a - a = a$$

So.. $$v = \frac{c^2(-a/2c)}{a} = \boxed{-\frac{1}{2}c}$$

K' moves in negative x-direction with respect to K at half the speed of light.

2-14 An event occurs in K' at $x'=2m$, $y'=3.5m$, $z'=3.5m$, and $t'=0$.
K' moves with speed $v=0.8c$ with respect to K.
What are the coordinates of the event in frame K?

Inverse Lorentz txn equations: $x=\gamma(x'+vt')$, $y=y'$, $z=z'$, $t=\gamma(t'+\frac{vx'}{c^2})$

$$\gamma=\frac{1}{\sqrt{1-(\frac{4}{5})^2}}=\frac{5}{3} \quad \text{so...} \quad \boxed{x=\frac{5}{3}\cdot 2\,m}=\frac{10}{3}\,m=\boxed{3.33\,m}$$

$$y=3.5m,\ z=3.5m$$

and $$t=\gamma\left(t'+\frac{vx'}{c^2}\right)=\frac{5}{3}\left(0+\underbrace{\frac{4}{5}}_{\frac{v}{c}}\cdot\frac{2m}{3\times10^8\,m/s}\right)$$

$$=\frac{4}{3}\cdot\frac{2}{3\times10^8}\,s=\frac{8}{9}\times10^{-8}\,s=\underline{0.89\times10^{-8}\,s}$$

$$\boxed{t=8.9\,ns}$$

$\boxed{2\text{-}15}$ A light signal is sent from the origin of a system K at $t=0$ to the point $x = 3\,m$, $y = 5\,m$, $z = 10\,m$.

(a) At what time is the signal received?

Since light travels at the same speed in all inertial frames, find the distance travelled and divide by c. Since the light signal starts at the origin...

$$L = \sqrt{x^2+y^2+z^2} = \sqrt{9m^2+25m^2+100m^2} = \sqrt{134m^2} = \underline{\underline{11.57m}}$$

The time required for light to travel this distance is...

$$t = \frac{L}{c} = \frac{11.57m}{3\times10^8 \, m/s} = \underline{\underline{38.6ns}}$$

(b) Find (x', y', z', t') for the receipt of the signal in K' moving at speed $0.8c$.

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{5}{3}$$

$$x' = \gamma(x-vt) = \frac{5}{3}\left(3m - \frac{4}{5}c\cdot\frac{L}{c}\right) = 5m - \frac{4}{3}\cdot 11.57m = \boxed{-10.43m}$$

$$y' = y = 5m$$

$$z' = z = 10m$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) = \frac{5}{3}\left[3.86\times10^{-8}s - \frac{4}{5}\cdot c\cdot\frac{3m}{c^2}\right]$$

$$= 6.43\times10^{-8}s - \frac{4}{3\times10^8 m/s} = 5.10\times10^{-8}s = \boxed{51.0\,ns}$$

(c) Verify that the light travelled at speed c in frame K'.

$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{\sqrt{x'^2+y'^2+z'^2}}{t'} = \frac{\sqrt{(10.43m)^2+(5m)^2+(10m)^2}}{5.10\times10^{-8}s} = \frac{15.29m}{5.1\times10^{-8}s}$$

$$= \underline{3.0\times10^8 \, m/s}$$

check ✓

$\boxed{2\text{-}20}$ Trip to a planet 20 l.y. away. How fast must a rocket go if the round trip is to take no longer than 40 years in time for the astronauts aboard?

Consider the outbound trip:

The astronauts experience the proper time interval $\Delta t_0 = 20$ yrs.

The earthbound clocks measure the dilated time ...

$$\Delta t = \gamma \, \Delta t_0$$

This time interval is related to the distance observed by earthbound observers...

$L_0 = v \Delta t$, where v is the speed of the rocket: $L_0 = 20$ l.y.

So ...

$$\frac{L_0}{v} = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} \qquad (1)$$

Solve for v : Inverting equation (1) : $\dfrac{v}{L_0} = \dfrac{\sqrt{1 - v^2/c^2}}{\Delta t_0}$

Or...

$$\frac{v^2}{L_0^2} = \frac{\left(1 - (v/c)^2\right)}{\Delta t_0^2}$$

Or...

$$\left(\frac{v}{c}\right)^2 \cdot \frac{(c\Delta t_0)^2}{L_0^2} = \left(1 - \left(\frac{v}{c}\right)^2\right)$$

$$\left(\frac{v}{c}\right)^2 \left[1 + \left(\frac{c\Delta t_0}{L_0}\right)^2\right] = 1$$

or...

$$\frac{v}{c} = \frac{1}{\sqrt{1 + \left(\frac{c\Delta t_0}{L_0}\right)^2}} = \frac{1}{\sqrt{\left(1 + \left(\frac{20 \text{ l.y.}}{20 \text{ l.y.}}\right)^2\right)}} = \frac{1}{\sqrt{2}}$$

or..

$$\boxed{v = 0.71\,c}$$

2-23 A clock in a spaceship is observed to run at a speed that is 3/5 that of a clock at rest on earth. How fast is the spaceship moving?

Use the time dilation formula: $\Delta t = \gamma \Delta t_0$

$\uparrow$ proper time is observed by the clock on the spaceship.

Algebra ... solve for v:

$$\left(\frac{\Delta t_0}{\Delta t}\right) = \frac{1}{\gamma}$$

$$\left(\frac{\Delta t_0}{\Delta t}\right) = \sqrt{1-\left(\frac{v}{c}\right)^2}$$

$$\left(\frac{\Delta t_0}{\Delta t}\right)^2 = 1-\left(\frac{v}{c}\right)^2$$

$$\frac{v}{c} = \sqrt{1-\left(\frac{\Delta t_0}{\Delta t}\right)^2}$$

$\frac{\Delta t_0}{\Delta t} = \frac{3}{5}$ from the statement of the problem, so...

$$\frac{v}{c} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} \Rightarrow \boxed{\frac{v}{c} = \frac{4}{5} = 0.8}$$

$\boxed{2\text{-}24}$ A spaceship of length 40m at rest is observed to be 20m long when in motion. How fast is it moving?

Use the length contraction formula : $L = L_0\sqrt{1-\left(\frac{v}{c}\right)^2}$

proper length is the length at rest : 40m

length measured when the object is in motion .

Solve for v :

$$\frac{L}{L_0} = \sqrt{1-\left(\frac{v}{c}\right)^2}$$

$$\left(\frac{L}{L_0}\right)^2 = 1-\left(\frac{v}{c}\right)^2$$

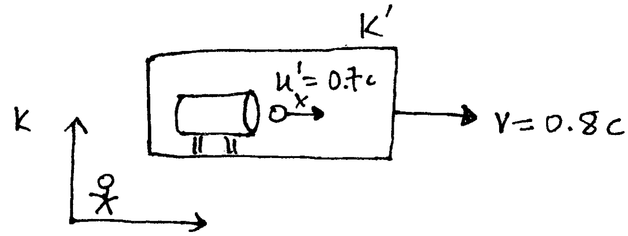
$$\frac{v}{c} = \sqrt{1-\left(\frac{L}{L_0}\right)^2}$$

$\frac{L}{L_0} = \frac{20m}{40m} = \frac{1}{2}$ from the statement of the problem..

$$\frac{v}{c} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2} \qquad \boxed{\frac{v}{c} = \frac{\sqrt{3}}{2} = 0.87}$$

2-31 Spaceship is moving at speed $V = 0.8c$ away from an observer. Proton gun on the spaceship fires protons at $0.7c$ (with respect to the spaceship).

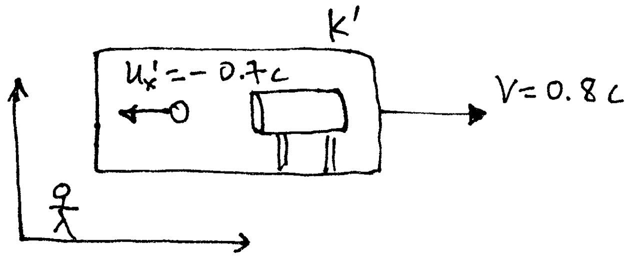
(a) If the proton gun if fired away from the observer at rest, what is the speed of the protons as observed by the observer at rest?



Use the inverse velocity addition equation: $u_x = \dfrac{u_x' + V}{1 + \frac{u_x' V}{c^2}}$

$$u_x = \frac{0.7c + 0.8c}{1 + \frac{(0.7)(0.8)c^2}{c^2}} = \frac{1.5c}{(1 + 0.56)} = \frac{1.5}{1.56}c = \boxed{0.962c}$$

(b) What speed does the stationary observer measure for the protons is the gun is fired toward the observer?

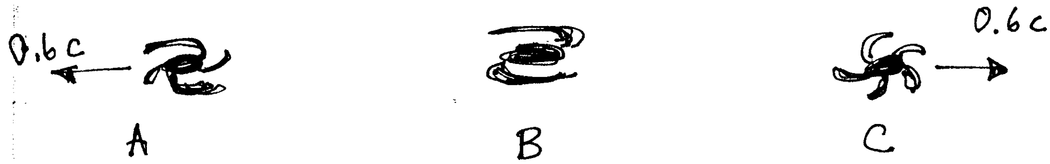


Use the inverse velocity addition formula $u_x = \dfrac{u_x' + V}{1 + \frac{u_x' V}{c^2}}$

$$u_x = \frac{-0.7c + 0.8c}{1 + \frac{(-.7)(.8)c^2}{c^2}} = \frac{0.1c}{(1 - 0.56)} = \frac{0.1c}{0.44} = \boxed{0.227c}$$

2-35 Three galaxies, A, B, and C.

Picture from frame B



What is the speed of galaxies B and C as viewed by someone in galaxy A?

Let B be frame K'. Then since A is moving at $v = -0.6c$ w.r.t to B, B is moving at $V = +\underline{0.6c}$ w.r.t. A.

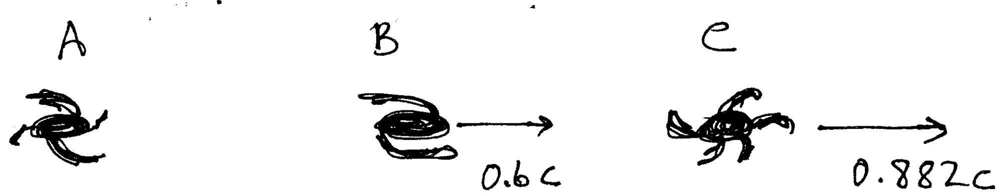
Let $u_x' = 0.6c$ be the velocity of C viewed from K' (B's frame).

Then, use the velocity addition equation to find the velocity of C viewed from K (A's frame)..

$$u_x = \frac{u_x' + V}{1 + \frac{u_x' V}{c^2}} = \frac{0.6c + 0.6c}{1 + \frac{(0.6)(0.6)c^2}{c^2}}$$

$$u_x = \frac{1.2c}{1+0.36} = \frac{1.2}{1.36}c = \underline{\underline{0.882}}$$

Picture from A's frame



$\boxed{2\text{-}95}$ Twin ~~paradox example~~ : Mary leaves earth on her 30^{th} birthday and travels to Alpha Centauri (4.3 l.y. away as measured in earth frame), and ~~returns~~ turns around immediately ~~upon~~ arrival to return home on her 52^{nd} birthday.

(a) How fast must her spaceship travel?

Mary measures the proper time interval Δt_0 between her departure and her arrival at Alpha Centauri. Since the entire trip takes her 22 years, the outbound leg takes $\Delta t_0 = 11$ years. Earthbound clocks measure

$$\Delta t = \gamma \Delta t_0 \text{ , the dilated time.}$$

And this time interval is related to the distance measured on earth ... $\Delta t = \frac{L}{v}$ so...

$$\frac{L}{v} = \gamma \Delta t_0 \qquad \text{solve for } v.$$

$$\frac{L}{v} = \frac{\Delta t_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \qquad \text{invert both sides ...}$$

$$\frac{v}{L} = \frac{\sqrt{1-(v/c)^2}}{\Delta t_0} \qquad \text{or... } \left(\frac{v}{c}\right) = \frac{L}{c\Delta t_0}\sqrt{1-(v/c)^2} \qquad \text{square both sides}$$

$$\left(\frac{v}{c}\right)^2 = \left(\frac{L}{c\Delta t_0}\right)^2\left(1-\left(\frac{v}{c}\right)^2\right) \qquad \text{algebra...}$$

$$\left(\frac{v}{c}\right)^2\left[1+\left(\frac{L}{c\Delta t_0}\right)^2\right] = \left(L/c\Delta t_0\right)^2$$

$$\boxed{\frac{v}{c} = \frac{(L/c\Delta t_0)}{\sqrt{1+(L/c\Delta t_0)^2}}} = \frac{4.3\text{ l.y.}/11\text{ l.y.}}{\sqrt{1+\left(\frac{4.3}{11}\right)^2}} = \frac{0.391}{\sqrt{1+(0.391)^2}} = \boxed{0.364}$$

(b) How old will Frank be when Mary returns?

Frank observes a time elapsed given by ...

$$\Delta t = \frac{(2L)}{v} \leftarrow \text{total, round-trip distance}$$

$$= \frac{8.6\ \text{l.y.}}{0.364c} = \underline{23.6\ \text{yrs}}$$

So Frank will be 53.6 years old when Mary (his twin) returns.

A. Show that the spatial distance between two points is invariant under Galilean transformation.

In Frame K, consider points (x_1, y_1, z_1) and (x_2, y_2, z_2)

The distance between these points is.. $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

In frame K', the Galilean transformation places the two points at...

$$x_1' = x_1 - vt \qquad x_2' = x_2 - vt$$
$$y_1' = y_1 \qquad y_2' = y_2$$
$$z_1' = z_1 \qquad z_2' = z_2$$

and $d' = \sqrt{(x_2'-x_1')^2 + (y_2'-y_1')^2 + (z_2'-z_1')^2}$ is the distance between the same two points...

Substituting in the Galilean txn equations...

$$d' = \sqrt{((x_2 - vt) - (x_1 - vt))^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

"vt" terms cancel out.

$$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} = d \qquad \text{QED}$$

The distance between points ① and ② is the same in both frames. It is <u>invariant</u> under Galilean transformation.

B. Convert the speed of light from meters per second to (a) mph, and (b) ft/ns.

(a) $c = \frac{3.00\times10^{8}\,m}{s} \cdot \frac{10^{2}\,cm}{m} \cdot \frac{1\,in}{2.54\,cm} \cdot \frac{1\,ft}{12\,in} = \underline{9.84\times10^{8}\,ft/s} \times \frac{1\,mi}{5280\,ft} \times \frac{3600\,s}{hr}$

$= 6.71\times10^{8}\,mph$

$\boxed{c = 67.1 \text{ million miles per hour.}}$

(b) $c = 9.84\times10^{8}\,ft/s \cdot 10^{-9}\,s/ns = 0.984\,ft/ns$ or... $\boxed{c \simeq 1\,ft/ns}$

C. The propagation of light at finite speeds means the things we observe with our eyes occurred in the past.

For events that are 10-25 ft away, the delay ~~[illegible]~~ is, using the speed calculated above... $T_{delay} = \frac{D}{c}$

$\underline{\underline{10-25\,ns}}$ This is the typical delay for things observed in a classroom.

For events that are 500-1000 ft away, such as the events on a playing field when viewed from the upper deck. In this case the delays are..

$\simeq 500 - 1000\,ns = \underline{\underline{0.5 - 1.0\,\mu s}}$