PHYSICS 160 Spring 2009

Problem Set #4

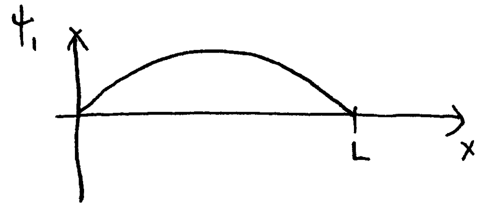
SOLUTIONS

$\boxed{6\text{-}17}$ Infinite square-well: Find the probability that a particle in its ground state is in each third of the one-dimensional box: $0 \le x \le L/3$, $L/3 \le x \le 2L/3$, and $2L/3 \le x \le L$.

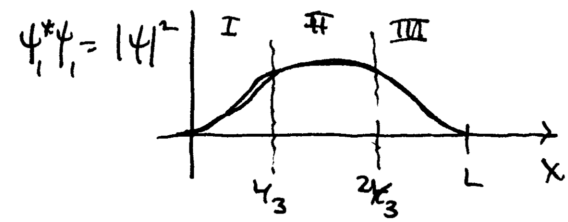
Wavefunction for particle in a 1D box: $\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ Eq. 6.34

↑ properly normalized

The ground state has $n=1$:



$\Rightarrow$



Clearly, by symmetry, the probability of finding the particle in the region $0 \le x \le L/3$ is the same as the probability of finding it in the region $2L/3 \le x \le L$. So, we only need to do two integrals

Regions I + III: $0 \le x \le L/3$ and $2L/3 \le x \le L$

$$P_I = P_{III} = \int_0^{L/3} \left(\frac{2}{L}\right) \sin^2\left(\frac{\pi x}{L}\right) dx$$

Use the trig. identity $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

$$P_I = P_{III} = \left(\frac{2}{L}\right)\cdot\frac{1}{2}\int_0^{L/3}\left(1 - \cos\left(\frac{2\pi x}{L}\right)\right)dx$$

$$= \frac{1}{L}\left[\int_0^{L/3} dx - \int_0^{L/3}\cos\left(\frac{2\pi x}{L}\right)dx\right]$$

$$= \frac{1}{L}\left[x\Big|_0^{L/3} - \frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\Big|_0^{L/3}\right]$$

$$= \frac{1}{L}\left[\frac{L}{3} - \frac{L}{2\pi}\sin\left(\frac{2\pi}{3}\right)\right] = \frac{1}{3} - \frac{1}{2\pi}\cdot\frac{\sqrt{3}}{2} = 0.1955 = \boxed{19.6\%}$$

Region II : $L/3 \le x \le 2L/3$

$$P_{II} = \int_{L/3}^{2L/3} \left(\frac{2}{L}\right) \sin^2\left(\frac{\pi x}{L}\right) dx$$

Use trig. identity $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

$$P_{II} = \left(\frac{2}{L}\right)\frac{1}{2}\int_{L/3}^{2L/3} \left(1 - \cos\left(\frac{2\pi x}{L}\right)\right) dx$$

$$= \frac{1}{L}\left[x \Big|_{L/3}^{2L/3} - \frac{L}{2\pi} \sin\left(\frac{2\pi x}{L}\right)\Big|_{L/3}^{2L/3} \right]$$

$$= \frac{1}{3} - \frac{1}{2\pi}\Big[\underbrace{\sin\left(\frac{4\pi}{3}\right)}_{-\frac{\sqrt{3}}{2}} - \underbrace{\sin\left(\frac{2\pi}{3}\right)}_{\frac{\sqrt{3}}{2}} \Big]$$

$$P_{II} = \frac{1}{3} + \frac{\sqrt{3}}{2\pi} = 0.609 = \boxed{60.9\%}$$

Check that $P_{tot} = P_I + P_{II} + P_{III} = 1$

$$= 2P_I + P_{II} = 2 \cdot 0.1955 + 0.609 = 1 \quad \text{check} \checkmark$$

$\boxed{6\text{-}20}$ What is the minimum energy of (a) a proton and (b) an α-particle trapped in a one-dimensional region the size of a uranium nucleus $(r = 7\times10^{-15}\,m)$?

Ground state energy of a particle in a 1D box: $E_1 = \dfrac{\pi^2\hbar^2}{2mL^2} = \dfrac{\pi^2(\hbar c)^2}{2(mc^2)L^2}$

(a) Proton rest mass energy $m_pc^2 = 938\text{ MeV} = 938\times10^6\text{ eV}$

So... $E_1 = \dfrac{\pi^2(197\text{ eV}\cdot\text{nm})^2}{2(938\times10^6\text{ eV})(7\times10^{-6}\text{ nm})^2} = \underline{4.17\times10^6\text{ eV}} = \boxed{4.17\text{ MeV}}$

convert to nm

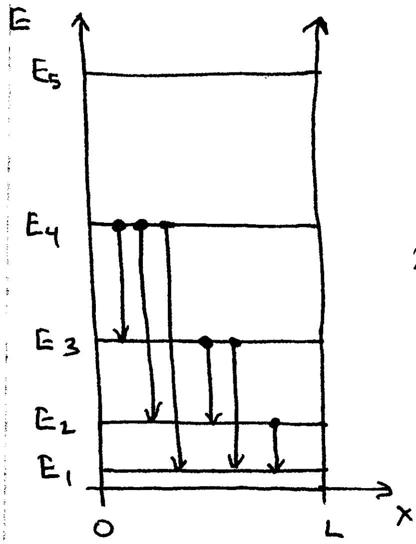
much less than rest mass energy

... justifies use of non-relativistic equations.

(b) α-particle rest mass energy $m_\alpha c^2 = 3727\text{ MeV} = 3.727\times10^9\text{ eV}$

So... $E_1 = \dfrac{\pi^2(197\text{ eV}\cdot\text{nm})^2}{2(3.727\times10^9\text{ eV})(7\times10^{-6}\text{ nm})^2} = 1.05\times10^6\text{ eV} = \boxed{1.05\text{ MeV}}$

6-21 An electron is trapped in an infinite square-well of width 0.5 nm. If the electron is initially in the n=4 state, what are the various photon energies that can be emitted as the electron jumps to the ground state?



$$E_n = E_1 n^2 \quad \text{where } E_1 = \frac{\pi^2 \hbar^2}{2mL^2} = \frac{\pi^2 (\hbar c)^2}{2(mc^2)L^2}$$

$n = 1, 2, 3, \ldots$

In this case $E_1 = \dfrac{\pi^2 (197\,\text{eV}\cdot\text{nm})^2}{2(511000\,\text{eV})(0.5\,\text{nm})^2} = \underline{\underline{1.5\,\text{eV}}}$

Photon energies from initial n=4 state....

$4 \to 3 \qquad \Delta E = E_1(4^2 - 3^2) = 7E_1 = \boxed{10.5\,\text{eV}}$

$4 \to 2 \qquad \Delta E = E_1(4^2 - 2^2) = 12E_1 = \boxed{18\,\text{eV}}$

$4 \to 1 \qquad \Delta E = E_1(4^2 - 1^2) = 15E_1 = \boxed{22.5\,\text{eV}}$

$3 \to 2 \qquad \Delta E = E_1(3^2 - 2^2) = 5E_1 = \boxed{7.5\,\text{eV}}$

$3 \to 1 \qquad \Delta E_1 = E_1(3^2 - 1^2) = 8E_1 = \boxed{12\,\text{eV}}$

$2 \to 1 \qquad \Delta E = E_1(2^2 - 1^2) = 3E_1 = \boxed{4.5\,\text{eV}}$

$\boxed{6-26}$ Find the energies of the second, third, fourth, and fifth levels for the 3D cubical box. Which are degenerate?

The energies for the 3D cubical infinite square well are... $E_{n_1n_2n_3} = \underbrace{\frac{\pi^2\hbar^2}{2mL^2}}(n_1^2+n_2^2+n_3^2)$

E_1 = ground state energy for 1D infinite square well.

Ground state: $E_{111} = E_1(1^2+1^2+1^2)$
$= 3E_1$ is NOT degenerate.

2nd energy level $n_1=2, n_2=1, n_3=1$ is three-fold degenerate...

$E_{211} = E_{121} = E_{112} = E_1(2^2+1^2+1^2)$
$= \boxed{6E_1}$

3rd energy level: $n_1=2, n_2=2, n_3=1$ is three-fold degenerate

$E_{221} = E_{212} = E_{122} = E_1(2^2+2^2+1^2)$
$= \boxed{9E_1}$

4th energy level: $n_1=3, n_2=1, n_3=1$ is three-fold degenerate

$E_{311} = E_{131} = E_{113} = E_1(3^2+1^2+1^2)$
$= \boxed{11E_1}$

5th energy level: $n_1=2, n_2=2, n_3=2$ is NOT degenerate

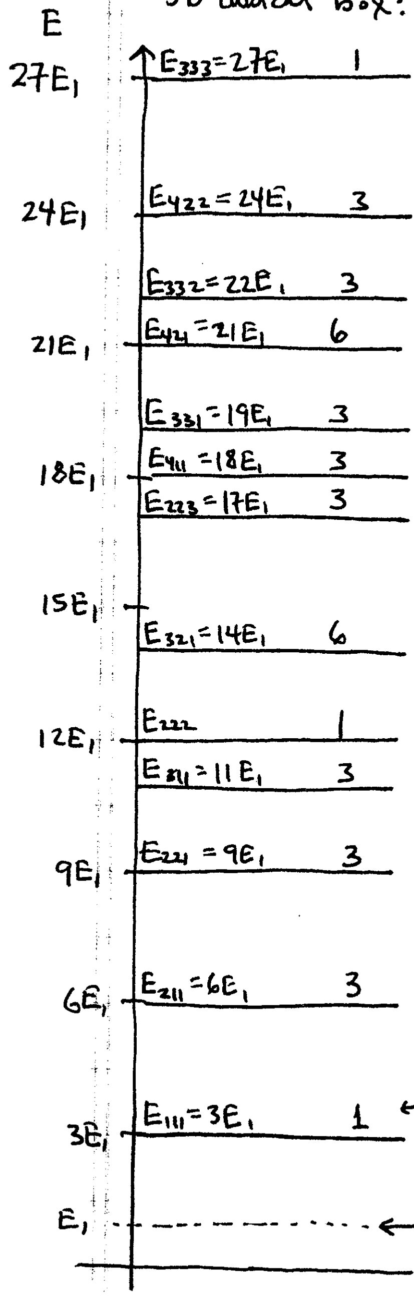
$E_{222} = E_1(2^2+2^2+2^2)$
$= \boxed{12E_1}$

Just for Fun.– here are first 13 energy levels and their degeneracy for the 3D cubical box:

If the quantum numbers are all equal, the state is not degenerate.

If one of the quantum numbers differs from the other two then the state is 3-fold degenerate.

If all three quantum numbers are different then the state is 6-fold degenerate.



$E_{333} = 27E_1$	1
$E_{422} = 24E_1$	3
$E_{332} = 22E_1$	3
$E_{421} = 21E_1$	6
$E_{331} = 19E_1$	3
$E_{411} = 18E_1$	3
$E_{223} = 17E_1$	3
$E_{321} = 14E_1$	6
E_{222}	1
$E_{311} = 11E_1$	3
$E_{221} = 9E_1$	3
$E_{211} = 6E_1$	3
$E_{111} = 3E_1$	1

$\leftarrow E_{321} = E_{312} = E_{231} = E_{213} = E_{132} = E_{123}$

$\leftarrow E_{211} = E_{121} = E_{112}$

← degeneracy

← ground state for 3D particle in a cube

← ground state energy for <u>1D</u> particle in a box

6-36

Show that the energy of a simple harmonic oscillator in the $n=1$ state is $3\hbar\omega/2$ by substituting the wave function $\psi_1 = Axe^{-\alpha x^2/2}$ directly into the Schrödinger Eq.

First recall that $\alpha = \sqrt{\frac{mK}{\hbar^2}}$ (Eq. 6.55a) where K is the spring constant

or... using the natural frequency of oscillation $\omega = \sqrt{\frac{K}{m}} \rightarrow \alpha = \frac{m\omega}{\hbar}$

$$\psi_1 = Axe^{-\frac{m\omega x^2}{2\hbar}}$$

Substitute this into the time-independent Schrödinger Eq. and show that it is a solution for $V(x) = \frac{1}{2}m\omega^2 x^2$, and $E = (n+\frac{1}{2})\hbar\omega$ with $n=1$.

Sch. Eq. $$-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V(x)\psi_1 = E_1\psi$$

Take derivatives... $$\frac{d\psi_1}{dx} = Ae^{-\frac{m\omega x^2}{2\hbar}} + Ax\underbrace{\left(-\frac{m\omega x}{\hbar}\right)}e^{-\frac{m\omega x^2}{2\hbar}}$$

derivative of argument of exponential

$$\frac{d}{dx}\left(-\frac{m\omega x^2}{2\hbar}\right) = -\frac{m\omega x}{\hbar}$$

$$\frac{d\psi_1}{dx} = Ae^{-\frac{m\omega x^2}{2\hbar}} - A\left(\frac{m\omega}{\hbar}\right)x^2 e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\frac{d^2\psi_1}{dx^2} = A\left(-\frac{m\omega x}{\hbar}\right)e^{-\frac{m\omega x^2}{2\hbar}} - A\left(\frac{2m\omega}{\hbar}\right)xe^{-\frac{m\omega x^2}{2\hbar}} - A\left(\frac{m\omega}{\hbar}\right)x^2\left(-\frac{m\omega x}{\hbar}\right)e^{-\frac{m\omega x^2}{2\hbar}}$$

$$= \left(-\frac{m\omega}{\hbar}\right)\underbrace{Axe^{-\frac{m\omega x^2}{2\hbar}}}_{\psi_1} - \left(\frac{2m\omega}{\hbar}\right)\underbrace{Axe^{-\frac{m\omega x^2}{2\hbar}}}_{\psi_1} + \left(\frac{m\omega}{\hbar}\right)^2 x^2 \underbrace{Axe^{-\frac{m\omega x^2}{2\hbar}}}_{\psi_1}$$

$$\boxed{\frac{d^2\psi_1}{dx^2} = \left[-\frac{3m\omega}{\hbar} + \left(\frac{m\omega}{\hbar}\right)^2 x^2\right]\psi_1}$$

Put this into the Sch. Eq. with $V(x) = \frac{1}{2}m\omega x^2$ — SHO potential energy function.

$$-\frac{\hbar^2}{2m}\left[-\frac{3m\omega}{\hbar}+\left(\frac{m\omega}{\hbar}\right)^2 x^2\right]\psi_1+\frac{1}{2}m\omega^2x^2\psi_1=E\psi_1$$

ψ_1 cancels out

$$E_1=\frac{3}{2}\hbar\omega - \frac{1}{2}m\omega^2x^2+\frac{1}{2}m\omega^2x^2$$

cancel

$\boxed{E_1=\frac{3}{2}\hbar\omega}$ which is the expected result for $n=1$.

So... we have verified that $\psi_1 = Axe^{-\frac{m\omega x^2}{2\hbar}}$ is a solution to the time-independent Schrödinger equation for the 1D simple harmonic oscillator, and that it corresponds to an energy of $\frac{3}{2}\hbar\omega$.

6-40 (a) Calculate the transmission probability for an α-particle of energy $E = 5$ MeV through a Coulomb barrier of a heavy nucleus that is approximated by a square barrier with $V_0 = 15$ MeV and width $L = 1.3 \times 10^{-14}$ m.

$E < V_0$ ... transmission probability is given by eq. 6.69

$$\boxed{T = \left[1 + \frac{V_0^2 \sinh^2(KL)}{4E(V_0 - E)}\right]^{-1}}$$

where $K = \dfrac{\sqrt{2m(V_0-E)}}{\hbar}$ or $\dfrac{\sqrt{2mc^2(V_0-E)}}{\hbar c}$

In this problem... $mc^2 = 3727$ MeV (rest mass energy of an α-particle)

So...

$$K = \frac{\sqrt{2(3.727\times10^9 \text{ eV})(10\times10^6 \text{ eV})}}{197 \text{ eV}\cdot\text{nm} \times 10^{-9} \text{ m/nm}} = 1.4\times10^{15} \text{ m}^{-1}$$

So... $KL = 1.4\times10^{15} \text{ m}^{-1} \cdot 1.3\times10^{-14} \text{ m} = \underline{18}$ this is substantially larger than unity, so we could use eq. 6.70...

$$T = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2KL}$$

$$T = 16\left(\frac{5}{15}\right)\left(1 - \frac{5}{15}\right)e^{-36} = 16\left(\frac{5}{15}\right)\left(\frac{10}{15}\right)e^{-36} = 16\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)e^{-36}$$

$$\boxed{T = 8.2\times10^{-16}}$$ This is a <u>tiny</u> <u>probability</u>

Use the more exact equation to confirm this result: Note that $\sinh(KL) = \frac{1}{2}(e^{KL} - e^{-KL})$

For $KL = 18$ ... $\sinh(KL) = \frac{1}{2}(e^{18} - e^{-18}) = 3.28\times10^7$

$$T = \left[1 + \frac{(15)^2 \cdot 1.08\times10^{15}}{4.5\cdot10}\right]^{-1} = 8.2\times10^{-16}$$ check ✓

(b) Also calculate the probability by doubling the barrier height. $V_0 = \underline{30\text{ MeV}}$

Recalculate K: $K = \dfrac{\sqrt{2mc^2(V_0 - E)}}{\hbar c} = \dfrac{\sqrt{2(3.727\times10^9\text{ eV})(25\times10^6\text{ eV})}}{197\text{ eV nm}\cdot 10^{-9}\text{ m/nm}}$

$K = 2.19\times10^{15}\text{ m}^{-1}$ So... $KL = 2.19\times10^{15}\text{ m}^{-1}\cdot 1.3\times10^{-14}\text{ m} = \underline{28}$

Since KL = 18 was sufficiently greater than unity to use eq. 6.70, we can use it again here..

$$T = 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2KL} = 16\left(\frac{5}{30}\right)\left(1-\frac{5}{30}\right)e^{-56}$$

$$T = 16\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)e^{-56} = 2.22e^{-56} = \boxed{1.06\times10^{-24}}$$

$\dfrac{T_b}{T_a} = \dfrac{1.06\times10^{-24}}{8.2\times10^{-16}} \doteq 10^{-9}$ doubling the barrier height reduces the transmission probability by a factor of 10^9 (1 billion)!

(c) Use the original barrier height, but double the width:

So......

$K = 1.4\times10^{15}\text{ m}^{-1}$ as in part (a) because the barrier width does not enter the equation for K.

$KL = 1.4\times10^{15}\text{ m}^{-1}\cdot 2.6\times10^{-14}\text{ m} = 36.4 \simeq \underline{36}$

$$T = 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2KL} = 16\left(\frac{5}{15}\right)\left(1-\frac{5}{15}\right)e^{-72}$$

$$= 16\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)e^{-72} = 3.55e^{-72} = \underline{1.91\times10^{-31}}$$

$\dfrac{T_c}{T_a} = \dfrac{1.91\times10^{-31}}{8.2\times10^{-16}} = 2\times10^{-16}$ doubling the barrier width reduces the transmission probability by a factor of $\sim 4\times10^{15}$!

6-43

Let 11.0eV electrons approach a potential barrier of height 3.8eV

(a) For what barrier thickness is there no reflection?

Note that $E > V_0$

From eq. 6.67, the transmission is 100% when $k_{II} L = n\pi$

$$\text{where } k_{II} = \frac{\sqrt{2m(E-V_0)}}{\hbar} = \frac{\sqrt{2mc^2(E-V_0)}}{\hbar c}$$

$$\text{In this case... } k_{II} = \frac{\sqrt{2(511{,}000\,\text{eV})(11.0\,\text{eV} - 3.8\,\text{eV})}}{197\,\text{eV}\cdot\text{nm}} = 13.77\ \text{nm}^{-1}$$

So... for barrier thickness ... $L = \frac{n\pi}{k_{II}} = n\left(\frac{\pi}{13.77\ \text{nm}^{-1}}\right) = n\,\underline{(0.228\ \text{nm})}$

transmission will be 100% ?

multiples of this this

$L = 0.228\ \text{nm},\ 0.456\ \text{nm},\ 0.684\ \text{nm},\ \ldots$

(b) For what barrier thickness is the reflection a maximum?

The maximum reflection occurs when T is a minimum, which happens when...

$$k_{II} L = \frac{n\pi}{2} \qquad \text{for } n = 1, 3, 5, 7, \ldots$$

So.. for $L = n\left(\frac{\pi}{2k_{II}}\right) = 0.114\ \text{nm},\ 0.342\ \text{nm},\ 0.570\ \text{nm},\ \ldots$

$\boxed{7-6}$ Show that the radial wave function ~~R~~ R_{21} for $n=2, l=1$ satisfies the radial differential equation (Eq. 7.10). What energy results? Is this consistent with the Bohr model?

$$R_{21}(r) = \frac{r}{a_0}\frac{e^{-r/2a_0}}{\sqrt{3}(2a_0)^{3/2}} \qquad n=2, l=1$$

Eq. 7.10
$$\overbrace{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}^{①} + \frac{2\mu}{\hbar^2}\left[E - V - \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\right]R = 0$$

~~where~~ where $V = \frac{-e^2}{4\pi\epsilon_0 r}$

① $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\left[\frac{e^{-r/2a_0}}{a_0\sqrt{3}(2a_0)^{3/2}} - \frac{r}{2a_0^2}\frac{e^{-r/2a_0}}{\sqrt{3}(2a_0)^{3/2}}\right]\right)$$

① $$\frac{1}{\sqrt{3}(2a_0)^{3/2}}\cdot\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2e^{-r/2a_0}}{a_0} - \frac{r^3e^{-r/2a_0}}{2a_0^2}\right]$$

① $$\frac{1}{\sqrt{3}(2a_0)^{3/2}}\cdot\frac{1}{r^2}\left[\frac{2re^{-r/2a_0}}{a_0} - \frac{r^2e^{-r/2a_0}}{2a_0^2} - \frac{3r^2e^{-r/2a_0}}{2a_0^2} + \frac{r^3e^{-r/2a_0}}{4a_0^3}\right]$$

① $$\underbrace{\frac{1}{\sqrt{3}(2a_0)^{3/2}}\left(\frac{r}{a_0}\right)e^{-r/2a_0}}\left[\frac{2}{r^2} - \frac{1}{2ra_0} - \frac{3}{2ra_0} + \frac{1}{4a_0^2}\right]$$

$$R_{21}(r)\left[\frac{2}{r^2} - \frac{4}{2ra_0} + \frac{1}{4a_0^2}\right]$$

Substitute this into Eq. 7.10 for $l=1$ $V = -e^2/4\pi\epsilon_0 r$

$$R_{21}(r)\left[\frac{2}{r^2} - \frac{2}{ra_0} + \frac{1}{4a_0^2}\right] + \frac{2\mu}{\hbar^2}\left[E + \frac{e^2}{4\pi\epsilon_0 r} - \frac{\hbar^2}{\mu r^2}\right]R_{21} = 0$$

$\div R_{21}(r)$ ....

$$\frac{2}{r^2} - \frac{2}{ra_0} + \frac{1}{4a_0^2} + \frac{2\mu E}{\hbar^2} + \frac{2\mu e^2}{4\pi\epsilon_0\hbar^2 r} - \frac{2}{r^2} = 0$$

cancel

Now recognize that $a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$; so.... the equation becomes

$$-\frac{2\mu e^2}{4\pi\epsilon_0\hbar^2 r} + \frac{\mu^2 e^4}{4(4\pi\epsilon_0)^2\hbar^4} + \frac{2\mu E}{\hbar^2} + \frac{2\mu e^2}{4\pi\epsilon_0\hbar^2 r} = 0$$

Those terms cancel.

Solve for E:

$$\frac{2\mu E}{\hbar^2} = -\frac{\mu^2 e^4}{4(4\pi\epsilon_0)^2\hbar^4}$$

$$E = \frac{-\mu e^4}{8(4\pi\epsilon_0)^2\hbar^2} = -\frac{E_0}{4} \leftarrow n^2 \text{ for } n=2$$

where $\underline{\underline{E_0 = \frac{\mu e^4}{2(4\pi\epsilon_0)^2\hbar^2}}} = \underline{\underline{13.6\,eV}}$ (see Eq. 4.26)

So R_{21} is indeed a solution to the radial equation for $n=2$, $\ell=1$, and gives the same energy as the Bohr model for $n=2$.

$\boxed{7\text{-}9}$ List all the possible quantum numbers for the n=6 level in atomic hydrogen.

$n=6,\ \ell=0 \qquad m_\ell = 0$

$\ell = 1 \qquad m_\ell = -1, 0, 1$

$\ell = 2 \qquad m_\ell = -2, -1, 0, 1, 2$

$\ell = 3 \qquad m_\ell = -3, -2, -1, 0, 1, 2, 3$

$\ell = 4 \qquad m_\ell = -4, -3, -2, -1, 0, 1, 2, 3, 4$

$\ell = 5 \qquad m_\ell = -5, -4, -3, -2, -1, 0, 1, 2, -3, 4, 5$

$\boxed{7\text{-}10}$ For a 3p state give the possible values of $n, \ell, m_\ell, L, L_z, L_x$, and L_y

The 3p notation indicates that $\underline{n=3}$, $\underline{\ell=1}$, $\boxed{L = \hbar\sqrt{\ell(\ell+1)} = \sqrt{2}\hbar}$

* $\boxed{m_\ell = -1}$ $\boxed{L_z = m_\ell\hbar = -\hbar}$,

The values of L_x and L_y are not "sharp" but must be consistent with... $L_x^2 + L_y^2 + L_z^2 = L^2$

So.. $L_x^2 + L_y^2 + \hbar^2 = 2\hbar^2$

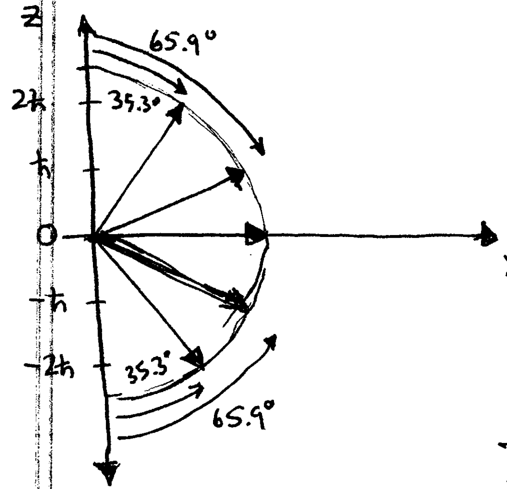
So.. $\boxed{L_x^2 + L_y^2 = \hbar^2}$

* $\boxed{m_\ell = 0}$ $\boxed{L_z = 0}$ $\boxed{L_x^2 + L_y^2 = 2\hbar^2}$

* $\boxed{m_\ell = 1}$ $\boxed{L_z = \hbar}$ $\boxed{L_x^2 + L_y^2 = \hbar^2}$

$\boxed{7\text{-}14}$ Draw for a 3d state all the possible orientations of the angular momentum vector $\vec{L}$.

The 3d state has $n=3$, $l=2$. So $m_l = -2, -1, 0, 1, 2$



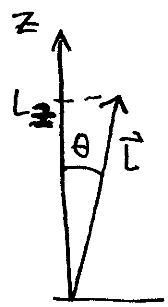
$$|\vec{L}| = \hbar\sqrt{l(l+1)} = \sqrt{6}\,\hbar$$

Angles of $\vec{L}$ with respect to the z-axis:

$$\theta = \cos^{-1}\left(\frac{L_z}{|\vec{L}|}\right) \qquad L_z = m_l\hbar$$

m_l	θ	$L_x^2 + L_y^2 = L^2 - L_z^2 = 6\hbar^2 - \hbar^2 m_l^2$
+2	35.3°	$2\hbar^2$
+1	65.9°	$5\hbar^2$
0	90°	$6\hbar^2$
−1	114.1°	$\boxed{5\hbar^2}$ ← answer to second part
−2	144.7°	$2\hbar^2$

$\boxed{7\text{-}15}$ What is the smallest value that l may have if $\vec{L}$ is within 3° of the z-axis?



$$\cos\theta = \frac{L_z}{L} = \frac{m_l\hbar}{\sqrt{l(l+1)}\,\hbar}$$

When L_z is closest to perfect alignment with the z-axis $m_l = l$

so.... $$\cos\theta = \frac{l}{\sqrt{l(l+1)}}$$

for $\theta = 3°$ $\quad \cos 3° = 0.9986 \ldots$

Or.. inverting equation $\quad \frac{\sqrt{l(l+1)}}{l} = (\cos\theta)^{-1} \qquad \frac{l(l+1)}{l^2} = (\cos^2\theta)^{-1}$

$$\frac{l^2 + l}{l^2} = \frac{1}{\cos^2\theta} \qquad \frac{1}{l} = \frac{1}{\cos^2\theta} - 1$$

$$1 + \frac{1}{l} = \frac{1}{\cos^2\theta} \qquad l = \left(\frac{1}{\cos^2\theta} - 1\right)^{-1} = \boxed{364}$$

minimum value for l to have $\vec{L}$ within 3° of z-axis