# The pro-$p$ Hom-form of the birational anabelian conjecture

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**Abstract**

We prove a pro-$p$ Hom-form of the birational anabelian conjecture for function fields over sub-$p$-adic fields. Our starting point is the corresponding Theorem of Mochizuki in the case of transcendence degree 1.

# 1 Introduction

## 1.1 Grothendieck's anabelian geometry

On June 27, 1983 A. Grothendieck wrote a letter [2] to G. Faltings in which he described his dream of an "anabelian" algebraic geometry, which has become known as the Grothendieck Conjecture:

> A general fundamental idea is that for certain, so-called "anabelian", schemes $X$ (of finite type) over $K$, the geometry of $X$ is completely determined by the (profinite) fundamental group $\pi_1(X, \xi) \ldots$ together with the extra structure given by the homomorphism:
>
> $$\pi_1(X, \xi) \to \pi_1(K, \xi) = \mathrm{Gal}(\overline{K}/K) \qquad \text{(p.280)}$$

The intuition that the arithmetic fundamental group of certain "anabelian" schemes over finitely generated fields, $K$, should be so extraordinarily rich as to completely specify the isomorphism type of the scheme has been borne out over the last quarter of a century, especially in the case of hyperbolic curves, which were the main class for which Grothendieck claimed the anabelian

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title [3], [4], [5], [11]. But Grothendieck also expected the function fields of algebraic $K$-varieties to be anabelian, giving rise to the birational version of the Grothendieck Conjecture which can be viewed as a generalization of the Galois characterization of global fields due to J. Neukirch, K. Uchida, et. al. [6], [12], [7], [8].

In the years since Grothendieck's letter, the anabelian philosophy has evolved to include various expectations about recovering schemes from quotients of the arithmetic fundamental group, such as the pro-$p$ situation considered here. Furthermore, it has become clear that one need not restrict oneself to finitely generated base fields, and in fact anabelian phenomena even occur over algebraically closed base fields, i.e. in the absence of any arithmetic structure [10]. Perhaps most importantly, Sh. Mochizuki discovered in [3] that

> ...the Grothendieck Conjecture for hyperbolic curves [is] an essentially local, $p$-adic result that belongs to that branch of arithmetic geometry known as $p$-adic Hodge theory. (p. 326)

Following Mochizuki, we are concerned here with the question of recovering embeddings of function fields over sub-$p$-adic fields from the induced homomorphisms of pro-$p$ Galois groups.

## 1.2 Statement of the Theorem

Let $k$ be a fixed sub-$p$-adic field, i.e., a subfield of a function field $k_1|\mathbb{Q}_p$. Fix $\overline{k}$, an algebraic closure of $k$, and denote by $G_k$ the absolute Galois group. Let $\mathcal{F}_k$ be the category of regular function fields $K|k$, and $k$-embeddings of function fields. Further, let $\mathcal{G}_k$ be the category of profinite groups, $G$, endowed with a surjective augmentation morphism, $\pi_G : G \to G_k$, such that $\ker(\pi_G)$ is a pro-$p$ group, and outer open $G_k$-homomorphisms, i.e., a morphism from $G$ to $H$ in $\mathcal{G}_k$ is of the form $\mathrm{Inn}_{G_k}(H) \circ f$, where $f : G \to H$ is an open homomorphism such that $\pi_G = \pi_H \circ f$, and $\mathrm{Inn}_{G_k}(H)$ denotes the group of inner automorphisms of $H$ which lie over $G_k$. Since $G_k$ has trivial center, $\mathrm{Inn}_{G_k}(H)$ consists exactly of the inner automorphisms of $H$ defined by an element of $\ker(\pi_H)$. Finally, we remark that there exists a naturally defined functor from $\mathcal{F}_k$ to $\mathcal{G}_k$: for $K|k$ from $\mathcal{F}_k$, let $\widetilde{K}|K\overline{k}$ be a maximal pro-$p$ extension. Then $\widetilde{K}|K$ is Galois, and $\Pi_K := \mathrm{Gal}(\widetilde{K}|K)$ endowed with the projection $\pi_K : \Pi_K \to G_k$ is an object of $\mathcal{G}_k$. Further, a morphism

$\imath : K|k \hookrightarrow L|k$ in $\mathcal{F}_k$ extends uniquely to a $\overline{k}$-embedding $\bar\imath : K\overline{k} \hookrightarrow L\overline{k}$, and $\bar\imath$ has prolongations $\tilde\imath : \widetilde{K} \hookrightarrow \widetilde{L}$. Each such prolongation $\tilde\imath : \widetilde{K} \hookrightarrow \widetilde{L}$ gives rise to an open $G_k$-homomorphism $\Phi_{\tilde\imath} : \Pi_L \to \Pi_K$ defined by

$$\Phi_{\tilde\imath}(g) = \tilde\imath^{-1} \circ g \circ \tilde\imath\,, \quad g \in \Pi_L\,,$$

and any two such prolongations are conjugate by an element from $\ker(\pi_K)$. Thus, sending each $K|k$ from $\mathcal{F}_k$ to $\pi_K : \Pi_K \to G_k$ in $\mathcal{G}_k$ yields a well defined functor from $\mathcal{F}_k$ to $\mathcal{G}_k$.

The purpose of this note is to prove the following *Galois by pro-p Hom-form* of the birational anabelian conjecture:

**Theorem 1.** *The above functor from $\mathcal{F}_k$ to $\mathcal{G}_k$ is fully faithful, i.e., for regular function fields $K|k$ and $L|k$, there is a canonical bijection*

$$\mathrm{Hom}_k(K, L) \to \mathrm{Hom}_{\mathcal{G}_\mathrm{k}}(\Pi_\mathrm{L}, \Pi_\mathrm{K}).$$

*Equivalently, for fixed field extensions $\widetilde{K}|K$ and $\widetilde{L}|L$ as above, the map*

$$\tilde\imath \mapsto \Phi_{\tilde\imath} \quad \textit{with} \;\; \Phi_{\tilde\imath}(g) = \tilde\imath^{-1} \circ g \circ \tilde\imath \;\; \textit{for} \;\; g \in \Pi_L\,,$$

*is a bijection from the set of $\overline{k}$-embeddings $\tilde\imath : \widetilde{K} \hookrightarrow \widetilde{L}$ onto the set of all the open $G_k$-morphisms $\Pi_L \to \Pi_K$.*

Before embarking on the proof, the following comments are in order: first, if $\mathrm{td}(K|k) = 1$, then Theorem 1 is a special case of Theorem 16.5 in the fundamental paper by Mochizuki [3]. Second, the above Theorem 1 implies the corresponding *full profinite version,* in which $\Pi_K$ is replaced by the full absolute Galois group $G_K$ of $K$. But naturally, the above Theorem 1 does not follow from the corresponding full profinite version. Finally, the full profinite version of Theorem 1 above was proved by Mochizuki in loc. cit., where it appears as Corollary 17.1. There he uses an inductive procedure on $\mathrm{td}(K|k)$ which is ill-suited to the pro-$p$ situation, and hence he obtains only a profinite result. In our proof of Theorem 1 we use Mochizuki's pro-$p$ result for the transcendence degree one case, but instead of proceeding inductively on the transcendence degree, we will make use of the second author's ideas as described in [9].

# 2 Proof of Theorem 1

The proof of Theorem 1 will have two parts:

i) Given an open $G_k$-homomorphism $\Phi : \Pi_L \to \Pi_K$, there exists a $\overline{k}$-embedding $\tilde{\imath}_\Phi : \widetilde{K} \hookrightarrow \widetilde{L}$ which defines $\Phi$ as indicated above, i.e., such that $\Phi = \Phi_{\tilde{\imath}_\Phi}$.

ii) The map $\tilde{\imath} \mapsto \Phi_{\tilde{\imath}}$ is injective.

First, let us recall the following basic facts about $p$-adic completions of abelian groups (which should not be confused with the pro-$p$ completions of such groups). For every abelian group $A$, let $\jmath_A : A \to \widehat{A} := \varprojlim_e A/p^e$ be the $p$-adic completion homomorphism from $A$ to its $p$-adic completion $\widehat{A}$. It is clear that the passage from $A$ to $\widehat{A}$ is a functor from the category of abelian groups (which is the same as the category of $\mathbb{Z}$-modules) to the category of $\mathbb{Z}_p$-modules. Moreover, $\ker(\jmath_A) = A_\infty := \{\, a \in A \mid \forall e > 0\ \exists b \in A \text{ s.t. } a = p^e b \,\}$ is the maximal $p^\infty$-divisible subgroup of $A$. In particular, if $0 \to A' \to A$ is a short exact sequence of abelian groups, then the resulting canonical sequence $0 \to \widehat{A'} \to \widehat{A}$ is exact if and only if $A'_\infty = A' \cap A_\infty$. Hence if $A/A'$ has trivial $p$-torsion, then $0 \to \widehat{A'} \to \widehat{A}$ is exact. In the sequel, we will consider/use the $p$-adic completions of the multiplicative groups $k^\times \subset K^\times$ of field extensions $K|k$ from $\mathcal{F}_k$, which we denote simply by $\widehat{k} := \widehat{k^\times}$, $\widehat{K} := \widehat{K^\times}$, etc.

We remark that for $K|k$ in $\mathcal{F}_k$ one has:

a) $K^\times/k^\times$ is a free abelian group, hence $\widehat{k} \hookrightarrow \widehat{K}$.

Indeed, this follows from the observation that $K^\times/k^\times$ is the subgroup of principal divisors inside the free abelian group $\mathrm{Div}(X)$, where $X$ is any projective normal model of the function field $K|k$.

b) $\ker(\jmath_K) = \mu'$ is the finite group of roots of 1 of order prime to $p$ in $k$.

Indeed, recall that $\ker(\jmath_K)$ equals the $p^\infty$-divisible subgroup of $K^\times$. Now since $K^\times/k^\times$ is a free abelian group, it follows that $\ker(\jmath_K) \subset k^\times$. And since $k$ is a sub-$p$-adic field, it is embeddable into a regular function field $k_1|k_0$, where $k_0|\mathbb{Q}_p$ is a finite field extension. But then reasoning as above, $k_1^\times/k_0^\times$ is a free abelian group, and so $\ker(\jmath_K) \subset k_0^\times$. Hence finally, $\ker(\jmath_K)$ is contained in the $p^\infty$-divisible subgroup of $k_0^\times$, which is the group $\mu'_{k_0}$ of roots of unity of order prime to $p$ in $k_0$, hence a finite group. Thus $\ker(\jmath_K) = \mu'_{k_0} \cap k =: \mu'$.

We next recall the following basic facts from Kummer Theory: For every $K|k$ from $\mathcal{F}_k$, let $\mathbb{T}_{p,k} = \varprojlim_e \mu_{p^e,\overline{k}}$ and $\mathbb{T}_{p,K} = \varprojlim_e \mu_{p^e,\widetilde{K}}$ be the Tate modules of $\mathbb{G}_{m,k}$, respectively $\mathbb{G}_{m,K}$. Then via the canonical inclusion $\bar\imath : \overline{k} \to \widetilde{K}$ we can/will identify $\mathbb{T}_{p,K}$ with $\mathbb{T}_p := \mathbb{T}_{p,k}$. Then Kummer Theory yields a canonical isomorphism of $p$-adically complete groups

$$\delta_K : \widehat{K} = \varprojlim_e K^\times/p^e \to \varprojlim_e \mathrm{H}^1(\Pi_K, \mu_{p^e,\widetilde{K}}) = \mathrm{H}^1(\Pi_K, \mathbb{T}_p).$$

Therefore we will make the identification $\widehat{K} = \mathrm{H}^1(\Pi_K, \mathbb{T}_p)$, if this does not lead to confusion. By the functoriality of Kummer Theory, the surjective projection $\pi_K : \Pi_K \to G_k$ gives rise to a canonical homomorphism $\mathrm{H}^1(\pi_K) : \widehat{k} \hookrightarrow \widehat{K}$ which is nothing but the $p$-adic completion of the structural morphism $k \hookrightarrow K$, and it is an embedding by remark a) above. Furthermore, if $K|k$ and $L|k$ are objects from $\mathcal{F}_k$, and $\Phi : \Pi_L \to \Pi_K$ is an open $G_k$-morphism, then by functoriality we get an embedding of $p$-adically complete groups

$$\mathrm{H}^1(\Phi) : \widehat{K} = \mathrm{H}^1(\Pi_K, \mathbb{T}_p) \to \mathrm{H}^1(\Pi_L, \mathbb{T}_p) = \widehat{L}$$

which identifies $\widehat{k} \subset \widehat{K}$ with $\widehat{k} \subset \widehat{L}$. Finally note that if $\Phi = \Phi_{\tilde\imath}$ is defined by a morphism $\imath : K|k \hookrightarrow L|k$ from $\mathcal{F}_k$, then $\mathrm{H}^1(\Phi_{\tilde\imath}) = \widehat{\imath}$ is nothing but the $p$-adic completion of the $k$-embedding $\imath : K|k \hookrightarrow L|k$, and therefore one has:

$$(\dagger) \qquad \mathrm{H}^1(\Phi_{\tilde\imath}) \circ \jmath_K(x) = \jmath_L \circ \imath(x), \quad x \in K^\times.$$

**Proof of i):**

*Claim 1.* $\mathrm{H}^1(\Phi) \circ \jmath_K(K^\times) \subseteq \jmath_L(L^\times)$.

*Proof of Claim 1*: Consider $t \in K^\times$ arbitrary. First, if $t \in k^\times$, then $\mathrm{H}^1(\Phi)$ identifies $\jmath_K(t)$ with $\jmath_L(t)$ by the discussion above. Second, let $t \in K^\times \backslash k^\times$. Then the inclusion $k(t) \subseteq K$ is a morphism in $\mathcal{F}_k$, hence gives rise canonically to a $\mathcal{G}_k$-morphism $\Phi_t : \Pi_K \to \Pi_{k(t)}$. But then $\Phi_t \circ \Phi : \Pi_L \to \Pi_{k(t)}$ is a $\mathcal{G}_k$-morphism too. Since $\mathrm{td}\big(k(t)|k\big) = 1$, it follows by Theorem 16.5 from [3] that $\Phi_t \circ \Phi$ is defined by a $k$-embedding $\imath_t : k(t) \to L$, i.e., is of the form $\Phi_t \circ \Phi = \Phi_{\tilde\imath_t}$. Hence by the assertion ($\dagger$) above, $\mathrm{H}^1(\Phi_t \circ \Phi) : \widehat{k(t)} \to \widehat{L}$ is exactly the $p$-adic completion of $\imath_t : k(t) \to L$, and we get:

$$\mathrm{H}^1(\Phi_t \circ \Phi) \circ \jmath_{k(t)}(x) = \jmath_L \circ \imath_t(x) \in \jmath_L(L^\times), \quad x \in k(t)^\times.$$

By functoriality, $\mathrm{H}^1(\Phi_t \circ \Phi) = \mathrm{H}^1(\Phi) \circ \mathrm{H}^1(\Phi_t)$, and $\mathrm{H}^1(\Phi_t)$ is the $p$-adic completion of the inclusion $k(t) \hookrightarrow K$. Hence $\mathrm{H}^1(\Phi_t) \circ \jmath_{k(t)}(x) = \jmath_K(x)$ for $x \in k(t)^\times$. Combining these equalities, we finally get

$$(\dagger') \qquad \mathrm{H}^1(\Phi) \circ \jmath_K(x) = \jmath_L \circ \imath_t(x) \in \jmath_L(L^\times), \quad x \in k(t)^\times.$$

Since $t$ was arbitrary, this concludes the proof of Claim 1. $\square$

Next let us identify $K^\times/\mu'$ and $L^\times/\mu'$ with their images in $\widehat{K}$, respectively $\widehat{L}$, via the $p$-adic completion homomorphisms $\jmath_K$, respectively $\jmath_L$. Then by Claim 1 above, $\mathrm{H}^1(\Phi)$ maps $K^\times/\mu'$ into $L^\times/\mu'$, and identifies $k^\times/\mu' \subset K^\times/\mu'$ with $k^\times/\mu' \subset L^\times/\mu'$. Further, $\mathrm{H}^1(\Phi)$ is nothing but the $p$-adic completion of its restriction to $K^\times/\mu'$. Modding out by $k^\times/\mu'$ thus yields an embedding of free abelian groups $\alpha : K^\times/k^\times \hookrightarrow L^\times/k^\times$ canonically defined by $\mathrm{H}^1(\Phi)$. Now we regard $(K,+)$ and $(L,+)$ as infinite dimensional $k$-vector spaces, and denote by $\mathcal{P}(K) := K^\times/k^\times$ and $\mathcal{P}(L) := L^\times/k^\times$ their projectivizations. Then $\alpha : \mathcal{P}(K) \hookrightarrow \mathcal{P}(L)$ is an inclusion which respects the multiplicative structures.

*Claim 2.* The map $\alpha : \mathcal{P}(K) \hookrightarrow \mathcal{P}(L)$ preserves lines.

*Proof of Claim 2*: A line in $\mathcal{P}(K)$ is the image of a two-dimensional $k$-subspace of $K$, say $\mathfrak{l}_{t_1t_2} := kt_1 + kt_2$, where $t_1, t_2 \in K$ are $k$-linearly independent. Note that $\mathfrak{l}_{t_1t_2} = t_1 \cdot \mathfrak{l}_t$, where $t = t_2/t_1$ and $\mathfrak{l}_t := k + kt$. Since multiplication by $t_1/k^\times$ is a line-preserving automorphism of $K^\times/k^\times = \mathcal{P}(K)$, $\alpha$ is multiplicative, and multiplication by $\alpha(t_1/k^\times)$ is a line-preserving automorphism of $\mathcal{P}(L)$, it suffices to show that $\alpha$ maps the lines $\mathfrak{l}_t$, $t \in K\backslash k$, to lines in $\mathcal{P}(L)$. In order to do this, we need only remark that by relation $(\dagger')$ above we have:

$$\mathrm{H}^1(\Phi) \circ \jmath_K(kt+k)^\times = \jmath_L \circ \imath_t(kt+k)^\times = \jmath_L\big(k\imath_t(t)+k\big)^\times.$$

Thus $\mathfrak{l}_t \subset \mathcal{P}(K)$ is mapped bijectively onto $\mathfrak{l}_{\imath_t(t)} \subset \mathcal{P}(L)$. $\square$

Let $L_0^\times \subseteq L^\times$ denote the preimage of $\alpha\big(\mathcal{P}(K)\big)$ in $L$, and let us set $L_0 := L_0^\times \cup \{0_L\} \subseteq L$. Since $\alpha$ preserves lines, it follows that $L_0$ is a field containing $k$, and $\alpha : \mathcal{P}(K) \to \mathcal{P}(L_0)$ is a line-preserving bijection. By the Fundamental Theorem of projective geometry (Theorem 2.26 of [1]), we conclude that $\alpha$ is induced by a $k$-semilinear isomorphism of $k$-vector spaces $\alpha_K : (K,+) \to (L_0,+)$, which is unique up to $k$-semilinear homotheties.

*Claim 3.* $\alpha_K$ is $k$-linear.

*Proof of Claim 3*: Let $\mu : k \to k$ be the field isomorphism with respect to which $\alpha_K$ is $k$-semilinear, i.e. $\alpha_K(ax) = \mu(a)\alpha_K(x)$ for all $a \in k, x \in K$. We wish to show that $\mu = \mathrm{id}_k$. For this, pick $t \in K^\times\backslash k^\times$ and consider the inclusion of $k$-vector spaces $(k(t),+) \subseteq (K,+)$. Then $\alpha_t := \alpha_K|_{k(t)}$ is a $k$-semilinear map with respect to $\mu$, with projectivization

$$\mathcal{P}(\alpha_t) = \mathcal{P}(\alpha_K)|_{\mathcal{P}(k(t))} = \alpha|_{\mathcal{P}(k(t))} = \mathcal{P}(i_t),$$

the last equality coming from relation $(\dagger')$ interpreted in terms of projectivizations. This immediately implies that $\mathrm{im}(i_t) \subseteq L_0$, so $i_t : k(t) \to L_0$ is a $k$-embedding a fields, *a fortiori* a $k$-linear map of $k$-vector spaces. By uniqueness, $i_t$ differs from $\alpha_t$ by a $k$-semilinear homothety, and in particular, $i_t$ must be $k$-semilinear with respect to $\mu$, so that that $\mu = \mathrm{id}_k$ as claimed. □

As in [9], it now follows that setting $\imath_K := \alpha_K(1_K)^{-1} \cdot \alpha_K$, the resulting map $\imath_K : K \to L$ is actually a $k$-isomorphism of fields, whose projectivization equals $\alpha$. In particular, the $p$-adic completion of $\imath_K$ equals $\mathrm{H}^1(\Phi)$, and $\imath_K$ is the unique embedding of fields $K \hookrightarrow L$ with this property.

Now let $K'|K$ be a finite Galois extension contained in $\widetilde{K}$, and $k' := K' \cap \overline{k}$. Then $\Pi_{K'}$ is an open normal subgroup of $\Pi_K$, and $\Phi^{-1}(\Pi_{K'}) = \Pi_{L'}$ for some finite Galois extension $L'|L$ contained in $\widetilde{L}$. Since $\Phi$ is a $G_k$-homomorphism, it follows that $L' \cap \overline{k} = k'$, and $K'|k'$ and $L'|k'$ are regular function fields over $k'$ which is a sub-$p$-adic field. The restriction $\Phi'$ of $\Phi$ to $\Pi_{L'}$ yields an open $G_{k'}$-homomorphism $\Phi' : \Pi_{L'} \to \Pi_{K'}$. *Mutatis mutandis,* we obtain from $\Phi'$ a $k'$-embedding $\imath_{K'} : K' \hookrightarrow L'$ such that its $p$-adic completion is $\mathrm{H}^1(\Phi')$. The compatibility relation $\mathrm{res}^{\Pi_{L'}}_{\Pi_L} \circ \mathrm{H}^1(\Phi') = \mathrm{H}^1(\Phi) \circ \mathrm{res}^{\Pi_{K'}}_{\Pi_K}$ translates into the fact that $\imath_K$ is the restriction of $\imath_{K'}$ to $K$. Note that extensions of the form $K'|K$ above exhaust $\widetilde{K}|K$, so taking limits we obtain a $\overline{k}$-embedding $\tilde{\imath}_\Phi : \widetilde{K} \hookrightarrow \widetilde{L}$ such that $(\tilde{\imath}_\Phi)|_{K'} = \imath_{K'}$ for all $K'|K$.

*Claim 4.* $\Phi_{\tilde{\imath}_\Phi} = \Phi$.

*Proof of Claim 4*: Indeed, for $K'|K$ and the corresponding $L'|L$ as above, $\Phi$ yields a surjection $\Psi' : \mathrm{Gal}(L'|L) \to \mathrm{Gal}(K'|K)$. Note that $\mathrm{H}^1(\Pi_{K'}, \mathbb{T}_p)$ is a $\mathrm{Gal}(K'|K)$ module, and correspondingly for $L'|L$. Moreover, for all $\sigma \in \mathrm{Gal}(L'|L)$ we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}^1(\Pi_{K'}, \mathbb{T}_p) & \xrightarrow{\mathrm{H}^1(\Phi')} & \mathrm{H}^1(\Pi_{L'}, \mathbb{T}_p) \\
{\scriptstyle \Psi'(\sigma)}\downarrow & & \downarrow{\scriptstyle \sigma} \\
\mathrm{H}^1(\Pi_{K'}, \mathbb{T}_p) & \xrightarrow{\mathrm{H}^1(\Phi')} & \mathrm{H}^1(\Pi_{L'}, \mathbb{T}_p).
\end{array}$$

which via the Kummer Theory isomorphisms translates into:

$$\imath_{K'} \circ \Psi'(\sigma) = \sigma \circ \imath_{K'}, \quad \sigma \in \mathrm{Gal}(K'|K).$$

Hence taking limits over all the $K'|K$, we get as required:

$$\tilde{\imath}_\Phi \circ \Phi(g) = g \circ \tilde{\imath}_\Phi, \quad g \in \Pi_L\,. \ \square$$

**Proof of ii):** Let $\tilde{\imath} : \widetilde{K} \hookrightarrow \widetilde{L}$ be a $\overline{k}$-embedding, and $\Phi_{\tilde{\imath}} : \Pi_L \to \Pi_K$ be the corresponding open $G_k$-homomorphism. For every finite Galois sub-extension $L'|L$ of $\widetilde{L}|L$ and the corresponding $K' := \tilde{\imath}^{-1}(L')$, the restriction of $\Phi_{\tilde{\imath}}$ to $\Pi_{L'}$ defines an open $G_{k'}$-homomorphism $\Phi_{\tilde{\imath},L'} : \Pi_{L'} \to \Pi_{K'}$, where $k' := \overline{k} \cap L'$. By the discussions above, the Kummer morphism $\mathrm{H}^1(\Phi_{\tilde{\imath},L'})$ is the $p$-adic completion of $\imath_{K'} := \tilde{\imath}|_{K'}$, and in particular, $\mathrm{H}^1(\Phi_{\tilde{\imath},L'})$ determines $\imath_{K'}$ uniquely. By taking limits, it follows that $\tilde{\imath}$ is uniquely determined by the family of Kummer morphisms $\mathrm{H}^1(\Phi_{\tilde{\imath},L'})$ with $L'|L$ as above. Assertion ii) is thus proven. □

This concludes the proof of Theorem 1.

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