GRAPH-THEORETIC HURWITZ GROUPS

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Abstract. This paper studies the analogue of Hurwitz groups and surfaces in the context of harmonic group actions on finite graphs. Our main result states that maximal graph groups are exactly the finite quotients of the modular group $\Gamma = \langle x, y \mid x^2 = y^3 = 1 \rangle$ of size at least 6. As an immediate consequence, every Hurwitz group is a maximal graph group, and the final section of the paper establishes a direct connection between maximal graphs and Hurwitz surfaces via a construction due to Brooks and Makover.

1. Introduction

Many recent papers have explored tantalizing analogies between Riemann surfaces and finite graphs (e.g. [2],[3],[4],[6],[7],[8],[11],[12]). Inspired by the Accola-Maclachlan [1], [15] and Hurwitz [13] genus bounds for holomorphic group actions on compact Riemann surfaces, we introduced harmonic group actions on finite graphs in [11], and established sharp linear genus bounds for the maximal size of such actions. As noted in the introduction to [11], it would be interesting to classify the groups and graphs that achieve the upper bound $6(g-1)$, thereby providing a graph-theoretic analogue of the study of Hurwitz groups and surfaces – those compact Riemann surfaces $\mathcal{S}$ of genus $g \geq 2$ such that $\mathrm{Aut}(\mathcal{S})$ has maximal size $84(g-1)$.

The investigation of Hurwitz groups has been a rich and active area of research, and much is known about their classification including a complete analysis of the 26 sporadic simple groups: 12 of them (including the Monster!) are Hurwitz, while the other 14 are not (see [9], [10] for an overview). The starting point for work on Hurwitz groups is the following generation result: a finite group G is a Hurwitz group if and only if it is a non-trivial quotient of the (2,3,7)-triangle group Δ with presentation

$$\Delta = \langle x, y \mid x^2 = y^3 = (xy)^7 = 1 \rangle .$$

That is: the Hurwitz groups are exactly the finite groups generated by an element of order 2 and an element of order 3 such that their product has order 7. The connection between the abstract group Δ and Hurwitz groups comes from the fact that Hurwitz surfaces arise as branched covers of the thrice-punctured Riemann sphere with special ramification. Such covers are nicely classified by the fundamental group of the punctured sphere, which is a free group on two generators.

The main result of this paper is an analogous generation result for *maximal graph groups* – those finite groups of size $6(g-1)$ that act harmonically on a finite graph of genus $g \geq 2$:

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I would like to thank my student Gus Black for determining the maximal graph groups that arise for graphs of low genus.

Theorem 1.1. *A finite group G is a maximal graph group if and only if $|G| \geq 6$ and G is a quotient of the modular group Γ with presentation $\Gamma = \langle x, y \mid x^2 = y^3 = 1 \rangle$.*

That is: the maximal graph groups are exactly the finite groups generated by an element of order 2 and an element of order 3. As an immediate and surprising corollary, we have:

Corollary 1.2. *Every Hurwitz group is a maximal graph group.*

As in the case of Hurwitz groups, the connection between the modular group Γ and maximal graph groups comes from the fact that maximal graphs occur as harmonic branched covers of trees (genus 0 graphs) with special ramification (Proposition 2.9). In order to classify such covers in general, we developed a harmonic Galois theory for finite graphs in [12], and the resulting concrete description of harmonic branched covers is the main tool used in the proof of Theorem 1.1, which we present in section 3. As preparation, we summarize the relevant background concerning harmonic group actions and Galois theory for finite graphs in section 2.

As mentioned above, the fact that every Hurwitz group is a maximal graph group comes as a surprise, because the relation between Riemann surfaces and finite graphs is largely analogical, rather than arising from a precise correspondence. However, there are a variety of direct connections between Riemann surfaces (and more generally algebraic curves) and finite graphs (see e.g. [2],[5],[6],[7],[16]). Of particular interest for us is the construction due to Brooks and Makover in [5] whereby a compact Riemann surface is associated to an oriented 3-regular graph. In the final section of this paper, we show that our theory meshes well with this construction in the following sense: if G is a maximal graph group, then G acts maximally on a 3-regular graph X. Moreover, the G-action endows X with an orientation, and the Brooks-Makover construction applied to X yields a compact Riemann surface $\mathcal{S}(X)$ on which G acts as a group of holomorphic automorphisms. Moreover, if G is actually a Hurwitz group, then the resulting surface $\mathcal{S}(X)$ is a Hurwitz surface with automorphism group G (Theorem 4.4). Thus, the Brooks-Makover construction provides a deep and unexpected link between the harmonic Galois theory of finite graphs and the Galois theory of Riemann surfaces.

2. Harmonic Group Actions

In this section, we briefly review some of the definitions and results from [4], [11], and [12]. To begin, by a *graph* we mean a finite multi-graph without loop edges: two vertices may be connected by multiple edges, but no vertex has an edge to itself. We denote the (finite) vertex-set of a graph X by $V(X)$, and the (finite) edge-set by $E(X)$. For a vertex $x \in V(X)$, we write $x(1)$ for the subgraph of X induced by the edges incident to x:

$$V(x(1)) := \{x\} \cup \{w \in V(X) \mid w \text{ is adjacent to } x\}$$
$$E(x(1)) := \{e \in E(X) \mid e \text{ is incident to } x\}.$$

The *genus* of a connected graph X is the rank of its first Betti homology group: $g(X) := |E(X)| - |V(X)| + 1$.

Definition 2.1. A *morphism of graphs* $\phi : Y \to X$ is a function $\phi : V(Y) \cup E(Y) \to V(X) \cup E(X)$ mapping vertices to vertices and such that for each edge $e \in E(Y)$ with endpoints $y_1 \neq y_2$, either $\phi(e) \in E(X)$ has endpoints $\phi(y_1) \neq \phi(y_2)$, or

$\phi(e) = \phi(y_1) = \phi(y_2) \in V(X)$. In the latter case, we say that the edge e is *ϕ-vertical*. ϕ is *degenerate* at $y \in V(Y)$ if $\phi(y(1)) = \{\phi(y)\}$, i.e. if ϕ collapses a neighborhood of y to a vertex of X. The morphism ϕ is *harmonic* if for all vertices $y \in V(Y)$, the quantity $|\phi^{-1}(e') \cap y(1)|$ is independent of the choice of edge $e' \in E(\phi(y)(1))$.

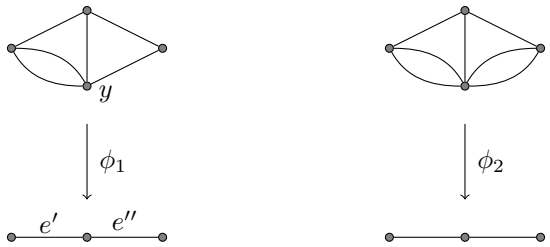

FIGURE 1. Each morphism is given by vertical projection. ϕ_1 is not harmonic at y, because the edge e' has two pre-images incident to y, while the edge e'' has only one. The morphism ϕ_2 is harmonic.

Definition 2.2. Let $\phi : Y \to X$ be a harmonic morphism between graphs, with X connected. If $|V(X)| > 1$ (i.e. if X is not the point graph $\star$), then the *degree* of the harmonic morphism ϕ is the number of pre-images in Y of any edge of X (this is well-defined by [4], Lemma 2.4). If $X = \star$ is the point graph, then the *degree* of ϕ is defined to be $|V(Y)|$, the number of vertices of Y.

Definition 2.3. Suppose that $G \leq \mathrm{Aut}(Y)$ is a (necessarily finite) group of automorphisms of the graph Y, so that we have a left action $G \times Y \to Y$ of G on Y. We say that (G, Y) is a *faithful group action* if the stabilizer of each connected component of Y acts faithfully on that component. Note that this condition is automatic if Y is connected.

Given a faithful group action (G, Y), we denote by $G\backslash Y$ the quotient graph with vertex-set $V(G\backslash Y) = G\backslash V(Y)$, and edge-set

$$E(G\backslash Y) = G\backslash E(Y) - \{Ge \mid e \text{ has endpoints } y_1, y_2 \text{ and } Gy_1 = Gy_2\}.$$

Thus, the vertices and edges of $G\backslash Y$ are the left G-orbits of the vertices and edges of Y, with any loop edges removed. There is a natural morphism $\phi_G : Y \to G\backslash Y$ sending each vertex and edge to its G-orbit, and such that edges of Y with endpoints in the same G-orbit are ϕ_G-vertical. As demonstrated in Figure 2, the quotient morphism ϕ_G is not necessarily harmonic, which motivates the following definition.

Definition 2.4. Suppose that (G, Y) is a faithful group action. Then (G, Y) is a *harmonic group action* if for all subgroups $H < G$, the quotient morphism $\phi_H : Y \to H\backslash Y$ is harmonic.

The condition in Definition 2.4 is quite restrictive, but the following proposition provides a simple criterion for harmonicity:

Proposition 2.5 ([12] Prop. 2.7; [11] Prop. 2.5)**.** *Suppose that (G, Y) is a faithful group action. Then (G, Y) is a harmonic group action if and only if for every vertex*

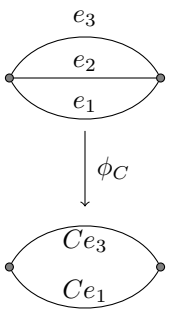

FIGURE 2. The cyclic group $C = \mathbb{Z}/2\mathbb{Z}$ acts faithfully on the upper graph by interchanging the edges e_1 and e_2 while fixing the edge e_3. The quotient morphism ϕ_C is not harmonic, because the edge of the quotient graph corresponding to the orbit Ce_3 has only one pre-image (the edge e_3), while the edge corresponding to Ce_1 has two preimages (e_1 and e_2).

$y \in V(Y)$, the stabilizer subgroup $I_y \leq G$ acts freely on the edge-set $E(y(1))$. Equivalently, (G, Y) is harmonic if and only if (after assigning an arbitrary direction to each edge of Y), the stabilizer subgroup of every directed edge is trivial.

By Proposition 2.5, if (G, Y) is a harmonic group action, then no directed edge of Y is fixed by a non-identity element of G, which implies that the stabilizers of (non-directed) edges of Y are either trivial or of order 2. That is: if the edge $e \in E(Y)$ is sent to itself by an element $\tau \in G$, then τ is an involution that switches the two endpoints of e. We refer to such an edge e as *flipped*, and if there are no flipped edges, then we say that the harmonic group action (G, Y) is *unflipped*. As explained in section 2 of [12], any harmonic group action (G, Y) has a unique *unflipped model*, obtained by replacing each flipped edge e with a pair of edges e, e' that are interchanged by the involution τ.

2.1. **Genus Bounds.** In [11], we established graph-analogues of the linear genus bounds for the maximal size of the automorphism group of a compact Riemann surface of genus $g \geq 2$. The situation for surfaces, as developed by Hurwitz [13], Accola [1], and Maclachlan [15], goes as follows. For each $g \geq 2$, define

$$N(g) := \max\{|\mathrm{Aut}(\mathcal{S})| \mid \mathcal{S} \text{ is a compact Riemann surface of genus } g\}.$$

Then $8(g+1) \leq N(g) \leq 84(g-1)$, and both of these bounds are sharp in the sense that the extreme values $8(g+1)$ and $84(g-1)$ are each attained infinitely often. $\mathcal{S}$ is called a *Hurwitz surface* if it attains the upper bound: $|\mathrm{Aut}(\mathcal{S})| = 84(g(\mathcal{S}) - 1)$. A finite group G is called a *Hurwitz group* if there exists a Hurwitz surface $\mathcal{S}$ with automorphism group isomorphic to G. The smallest Hurwitz group is $PSL_2(\mathbb{F}_7)$ which occurs in genus 3 as the automorphism group of Klein's quartic curve defined in homogeneous coordinates by the equation

$$x^3y + y^3z + z^3x = 0.$$

We now describe the graph-theoretic versions of these results from [11]. For each $g \geq 2$, define

$$M(g) := \max\{|G| \mid G \text{ acts harmonically on a connected graph of genus } g\}.$$

Then $4(g-1) \leq M(g) \leq 6(g-1)$, and these bounds are sharp in the sense that the extreme values $4(g-1)$ and $6(g-1)$ are each attained infinitely often. Moreover, unlike the case of Riemann surfaces, these two extremes are actually the *only* values taken by the function $M(g)$. A connected graph Y is called a *maximal graph* if it attains the upper bound, i.e. if there exists a finite group G acting harmonically on Y with $|G| = 6(g(Y)-1)$. In this case we call G a *maximal graph group* and say that G acts *maximally* on Y. The smallest maximal graph group occurs already in genus 2. In fact both groups of order $6 = 6(2-1)$ are maximal graph groups: the symmetric group $\mathfrak{S}_3$ and the cyclic group $\mathbb{Z}/6\mathbb{Z}$ each act maximally on the genus 2 graph consisting of 2 vertices connected by 3 edges.

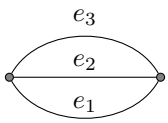

FIGURE 3. A generator of the cyclic group $\mathbb{Z}/6\mathbb{Z}$ acts by interchanging the two vertices while cyclically permuting the three edges. The symmetric group $\mathfrak{S}_3 = \langle \tau, \sigma \mid \tau^2 = \sigma^3 = 1, \sigma\tau = \tau\sigma^{-1} \rangle$ acts as follows: σ cyclically permutes the three edges, and τ interchanges the two vertices, flipping e_1 while interchanging e_2 and e_3.

2.2. **Harmonic Galois Theory.** In [12], we constructed a harmonic Galois theory for finite graphs with the goal of answering the following general question: if we fix a connected base graph X, how can we classify the connected harmonic G-covers $\phi : Y \to X$? By a harmonic G-cover $\phi : Y \to X$, we mean a harmonic group action (G, Y) together with an isomorphism $\overline{\phi} : G\backslash Y \tilde{\to} X$. Composing the isomorphism $\overline{\phi}$ with the quotient morphism ϕ_G then yields a harmonic morphism $\phi := \overline{\phi} \circ \phi_G$ from Y to X. In order to explain the classification, we need to introduce the following definitions, which are motivated by the Galois theory of algebraic curves defined over non-algebraically closed fields.

Definition 2.6. Suppose that $\phi : Y \to X$ is a harmonic G-cover of X, and $y \in V(Y)$ has image $x := \phi(y)$. The *decomposition group* $\Delta_y \leq G$ at y is the stabilizer of the connected component of the fiber $Y_x := \phi^{-1}(x)$ containing y. The *inertia group* I_y at y is the stabilizer subgroup of y in G. Note that $I_y \leq \Delta_y$, and the decomposition / inertia groups form conjugacy classes in G as y varies over the fiber Y_x. We say that ϕ is *horizontally unramified* or *étale at* y if $I_y = \{\varepsilon\}$, and the cover φ is *étale* if it is étale at all $y \in V(Y)$. The *horizontal ramification index* at y is $m_y := |I_y|$, and the *inertia degree* at y is $f_y := \#\{\text{vertices in the connected component of } Y_x \text{ containing } y\}$. Finally, the *vertical multiplicity* at y is $v_y := \#\{\phi\text{-vertical edges incident to } y\}$. Since all edges of the fiber Y_x are ϕ-vertical, v_y is the degree of the vertex y in the graph Y_x. Since by Proposition 2.5 the inertia group I_y acts freely on the edges incident to y, we see that the vertical multiplicity satisfies $v_y = m_y w_y$ for some $w_y \geq 0$. Note that the numbers m_y, f_y, v_y, and w_y are independent of the vertex y, and only depend on the image vertex $x = \phi(y)$. The *branch locus* of ϕ is the set of vertices $B \subset V(X)$ for which the corresponding fibers have either $m > 1$ or $v > 0$.

As evidence that these definitions are good analogues of their algebro-geometric / number-theoretic counterparts, we prove the following graph-theoretic version of the Fundamental Identity for primes in Galois extensions of global fields (see e.g. [18] Prop. 8.2).

Proposition 2.7. *Suppose that $\phi : Y \to X$ is a harmonic G-cover, and $y \in V(Y)$ with $x := \phi(y)$. Let n be the number of connected components of the fiber Y_x, and m, f be the ramification index and inertia degree at points of the fiber respectively. Then* $\deg(\phi) = mfn$.

Proof. We have the equalities $\deg(\phi) = |G| = |G/\Delta_y||\Delta_y/I_y||I_y|$. Since G acts transitively on the set of connected components of Y_x, the orbit stabilizer theorem yields $n = |G/\Delta_y|$. Similarly, since Δ_y acts transitively on the vertices of the connected component of Y_x containing y, we see that $f = f_y = |\Delta_y/I_y|$. Putting these observations together with the definition $m = m_y := |I_y|$ yields the Fundamental Identity. □

This Fundamental Identity for graphs provides further justification for our proposal in section 2 of [12] to interpret ϕ-vertical edges not as "vertical ramification" as in [4], but rather as the graph-theoretic analogue of an extension of residue fields.

An important tool for our study of maximal graph groups is the graph-analogue of the Riemann-Hurwitz formula established in [4], which we state here in the special case of harmonic G-covers as reformulated in section 2 of [11]:

Proposition 2.8 ([4] Theorem 2.14)**.** *Suppose that $\phi : Y \to X$ is a connected harmonic G-cover. Then*

$$2g(Y) - 2 = |G|(2g(X) - 2 - R),$$

where the ramification number $R := \sum_{x \in V(X)}[2(1 - \frac{1}{m_x}) + w_x]$. Here $m_x := m_y$ and $w_x := w_y = \frac{v_y}{m_y}$ for any choice of $y \in V(Y_x)$.

In section 5 of [11], we used the graph-theoretic Riemann-Hurwitz formula to show that if (G, Y) is a maximal harmonic G-action, then the quotient $G\backslash Y$ is a tree, and the ramification number for the quotient morphism $Y \to G\backslash Y$ is $R = \frac{7}{3}$. Moreover, an earlier proposition from [11] shows that $R = \frac{7}{3}$ can only occur in three ways:

Proposition 2.9 ([11] Prop. 3.3 and section 5)**.** *Suppose that $\phi : Y \to X$ is a connected harmonic G-cover. Then ϕ is maximal ($|G| = 6(g(Y) - 1)$) if and only if X is a tree and the ramification number for ϕ is $R = \frac{7}{3}$. In this case, there are exactly three possibilities for the branch locus $B \subset V(X)$, up to a reordering of the branch points:*

(i) *a single branch point with $m = 3, w = 1$;*
(ii) *two branch points with ramification vector $(m_1, m_2; w_1, w_2) = (3, 2; 0, 0)$;*
(iii) *two branch points with ramification vector $(m_1, m_2; w_1, w_2) = (3, 1; 0, 1)$.*

In section 3 of [12], we showed that étale G-covers of X are classified by a certain group (called the *étale fundamental group of X*) which is isomorphic to the free profinite completion of the free group on countably many generators. In section 4 of *loc. cit.* we gave a more concrete description of this result. Since we will only need to use this description in the case where X is a tree, we content ourselves with describing that case here: to give an unflipped étale G-cover of a tree X, we just need to specify, for each vertex of X, a finite, symmetric, and unordered multi-set

of non-trivial elements of G. By a multi-set, we mean that the elements of G may appear with multiplicity, and by symmetric we mean that if the element ρ appears, then ρ^{-1} also appears with the same multiplicity. If S is such a multi-set, we may construct the Cayley graph $\mathrm{Cay}(G, S)$ with vertex set G as follows (see [12], Example 2.8): for each vertex $g \in G$ and for each $\rho \in S$, there is an edge from g to $g\rho$. Furthermore, if $\rho \neq \rho^{-1}$, then the ρ-edge from g to $g\rho$ is identified with the ρ^{-1}-edge from $g\rho$ to $g = g\rho\rho^{-1}$. Edges coming from involutions in S are not identified in this fashion. Inverse-pairs of group elements appearing with multiplicity in S yield multiple edges of $\mathrm{Cay}(G, S)$. The resulting Cayley graph supports a natural unflipped harmonic G-action given by left-multiplication on the vertex labels in G. We associate to each vertex of X the Cayley graph constructed from the given multi-set; these form the fibers of the corresponding unflipped étale G-cover, and they are glued together according to the tree X. The union of the multi-sets must generate the group G in order for the resulting G-cover to be connected.

We illustrate this construction for the symmetric group $G = \mathfrak{S}_3$ and X the graph consisting of two vertices x_1 and x_2 connected by a single edge e. We have the presentation

$$\mathfrak{S}_3 = \left\langle \tau, \sigma \mid \tau^2 = \sigma^3 = 1, \sigma\tau = \tau\sigma^{-1} \right\rangle.$$

Choose $S_1 = \{\sigma, \sigma^{-1}\}$ and $S_2 = \{\tau\}$ for the symmetric multi-sets corresponding to x_1 and x_2 respectively. Their union generates $\mathfrak{S}_3$, so the corresponding étale G-cover $Y^{\text{ét}}$ will be connected. The fiber $Y^{\text{ét}}_{x_i}$ over the vertex x_i is given by the (disconnected) Cayley graph $\mathrm{Cay}(\mathfrak{S}_3, S_i)$, and vertices labeled by the same group element in the two fibers are connected by an edge lying over e. The group $\mathfrak{S}_3$ acts harmonically on $Y^{\text{ét}}$ via left-multiplication on the group elements labeling the vertices.

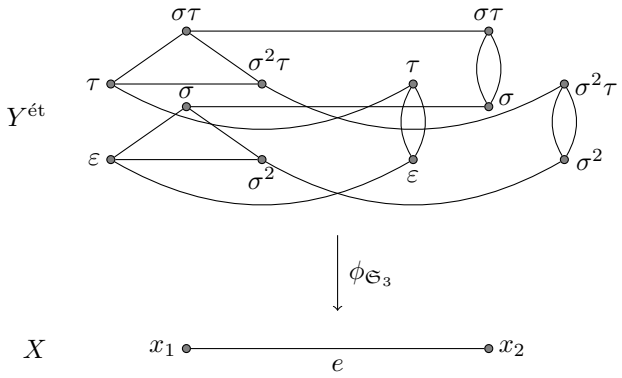

FIGURE 4. The unflipped étale $\mathfrak{S}_3$-cover of X corresponding to the symmetric multi-sets $S_1 = \{\sigma, \sigma^{-1}\}$ and $S_2 = \{\tau\}$.

The harmonic $\mathfrak{S}_3$-cover constructed above has no flipped edges. But the pairs of vertical edges corresponding to τ in the fiber $Y^{\text{ét}}_{x_2}$ may each be identified to a single flipped edge, thereby obtaining a harmonic $\mathfrak{S}_3$-cover $\overline{Y}^{\text{ét}} \to X$ whose unflipped model is $Y^{\text{ét}}$ (see Figure 5 and [12] section 2).

To allow for horizontal ramification, we introduced in [12] the notion of a *G-inertia structure* on the base X, which is simply a collection of subgroups indexed

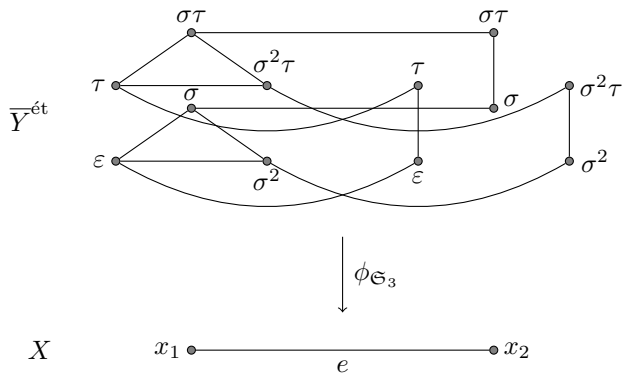

FIGURE 5. The étale $\mathfrak{S}_3$-cover of X with flipped edges corresponding to the symmetric multi-sets $S_1 = \{\sigma, \sigma^{-1}\}$ and $S_2 = \{\tau\}$.

by the vertices of X. If $\mathcal{I} = \{I_x \leq G \mid x \in V(X)\}$ is such a G-inertia structure on X, then there is a functor $\mathcal{F}^{\mathcal{I}}$ from the category of étale G-covers of X to the category of harmonic G-covers of X with inertia groups given by the conjugacy classes $C(\mathcal{I}) := \{c(I_x) \mid x \in V(X)\}$. The functor acts on each fiber by collapsing the vertex set G of the Cayley graph over x onto the set of left cosets G/I_x, removing any loop edges that are produced. In Proposition 5.2 of [12], we prove that every harmonic G-cover of X with inertia given by $C(\mathcal{I})$ arises via this construction. Thus, every harmonic G-cover of X can be described by specifying a G-inertia structure on X, together with a finite, symmetric, unordered multi-set of non-trivial elements of G for each vertex of X. The cover will be connected exactly when G is generated by the union of the inertia groups and the multi-sets.

Returning to our $\mathfrak{S}_3$-example, choose the inertia structure $\mathcal{I} = \{I_1, I_2\}$ with $I_1 = \langle\sigma\rangle$ and I_2 trivial. Then applying the functor $\mathcal{F}^{\mathcal{I}}$ to the $\mathfrak{S}_3$-cover $\overline{Y}^{\text{ét}} \to X$ has the following effect: the fiber $\overline{Y}^{\text{ét}}_{x_2}$ is unchanged, while the fiber $\overline{Y}^{\text{ét}}_{x_1} = \text{Cay}(\mathfrak{S}_3, S_1)$ is altered by collapsing the vertices onto the two left-cosets of I_1 in $\mathfrak{S}_3$ and removing the loop-edges that result (see Figure 6).

3. Proof of Theorem 1.1

In this section, we use the results of section 2.2 to prove

Theorem 1.1. *A finite group G is a maximal graph group if and only if $|G| \geq 6$ and G is a quotient of the modular group Γ with presentation $\Gamma = \langle x, y \mid x^2 = y^3 = 1\rangle$.*

Proof. ($\Longleftarrow$) Suppose that $|G| \geq 6$ and $\pi : \Gamma \to G$ is a surjection. Set $\tau := \pi(x), \sigma := \pi(y)$, so that τ has order 2 and σ has order 3 in G. Let $X = \star$ be the point graph. In order to construct a harmonic G-cover of X, we just need to specify a symmetric multi-set S together with an inertia group $I < G$. For this, we take $S = \{\tau\}$ and $I = \langle\sigma\rangle$, and define $Y := \mathcal{F}^I(\text{Cay}(G, S))$. Then $Y \to X$ is a harmonic G-cover of X, with inertia groups given by the conjugacy class of I in G. Moreover, Y is connected since G is generated by $I \cup S$. By construction, this is an unflipped action, but each pair of edges corresponding to τ may be replaced by a single flipped edge to obtain a connected harmonic G-cover $\overline{Y} \to X$. Moreover, since $|I| = 3$, every point of $\overline{Y}$ has inertia group of order 3 and is incident to 3 vertical edges.

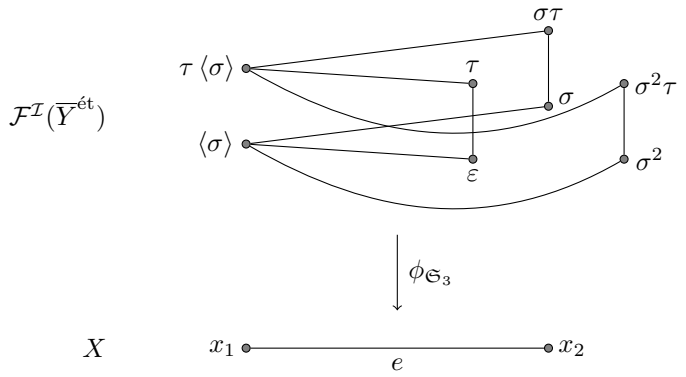

FIGURE 6. The harmonic $\mathfrak{S}_3$-cover of X with flipped edges corresponding to the symmetric multi-sets $S_1 = \{\sigma, \sigma^{-1}\}, S_2 = \{\tau\}$, and $\mathfrak{S}_3$-inertia structure $I_1 = \langle\sigma\rangle, I_2 = \{\epsilon\}$.

That is, the ramification of $\overline{Y} \to X$ corresponds to case (i) of Proposition 2.9 – a single branch point $\star$ with $m = 3$ and $w = 1$. It follows that the ramification number $R = \frac{7}{3}$, so that $|G| = 6(g(\overline{Y}) - 1)$, and G is a maximal graph group.

($\Longrightarrow$) Now suppose that G is a maximal graph group, so there exists a connected harmonic G-cover $\overline{Y} \to X$ where the genus of $\overline{Y}$ satisfies $|G| = 6(g(\overline{Y}) - 1) \geq 6$. Moreover, from [11] we know that X is a tree, and one of the three branch loci described in Proposition 2.9 occurs. We first consider case (i) of a single branch point with $m = 3$ and $w = 1$.

From the construction in [12], we may assume that $X = \star$ is the point graph, since the part of the tree outside of the single branch point is inessential in this case. Thus, the cover $\overline{Y} \to X$ is totally degenerate, and its unflipped model Y may be obtained as $\mathcal{F}^I(\mathrm{Cay}(G, S))$ for some inertia group $I < G$ and symmetric multi-set S of elements from G. Fix such a choice of I and S, where we may assume that $I \cap S = \emptyset$. (This is because any edge of the Cayley graph coming from the intersection will be removed as a loop edge when we apply $\mathcal{F}^I$.) Since $m = 3$, we must have $I \cong \mathbb{Z}/3\mathbb{Z}$; choose a generator σ for this inertia subgroup. The condition $w = 1$ means that each vertex of $\overline{Y}$ is incident to $m = 3$ vertical edges, and since the G-action is totally degenerate, we see that $\overline{Y}$ is in fact 3-regular. This is only possible if every edge of $\overline{Y}$ is flipped, having been obtained from a pair of edges in the unflipped model Y. Indeed, since $\mathrm{Cay}(G, S)$ has degree at least 2, and the functor $\mathcal{F}^I$ identifies 3 vertices in $\mathrm{Cay}(G, S)$ to a single vertex in Y, we see that $Y = \mathcal{F}^I(\mathrm{Cay}(G, S))$ has degree at least 6 (here we use the fact that $I \cap S = \emptyset$). Hence, $\mathrm{Cay}(G, S)$ must be 2-regular, with the property that the edges of $\mathcal{F}^I(\mathrm{Cay}(G, S))$ may be identified in pairs to produce $\overline{Y}$. This leads to two possibilities for the multi-set S: either $S = \{\tau\}$ where τ has order 2, or $S = \{\rho, \rho^{-1}\}$ where $\rho \in \tau I$ for some element τ of order 2. The second option requires some explanation: if the element ρ is to yield a flipped edge of $\overline{Y}$, then it must connect two vertices that are interchanged by an element $\tau \in G$ of order 2. The vertices of $\overline{Y}$ are labeled by the left cosets of I, so the edge corresponding to ρ must connect I and τI. But the element ρ yields an edge of $\overline{Y}$ connecting I to ρI, so it follows that $\rho I = \tau I$, which is equivalent to the stated condition $\rho \in \tau I$.

Since $\overline{Y}$ is connected, we see that G is generated by $I \cup S$. In both of the cases described above, this implies that G is generated by τ and σ. Hence, we may define a surjection $\pi : \Gamma \to G$ by $\pi(x) = \tau$ and $\pi(y) = \sigma$, showing that G is a quotient of Γ as required.

Now assume that we are in case (ii) or (iii) of Proposition 2.9: two branch points x_1 and x_2 with ramification vector $(m_1, m_2; w_1, w_2) = (3, 2; 0, 0)$ or $(3, 1; 0, 1)$. Let P be the unique path between x_1 and x_2 in the tree X. Then as a first simplification, we may assume that $X = P$, since the part of the tree outside of P plays no essential role in the constructions from [12]. Going further, we may assume that X is a single edge connecting x_1 to x_2, since any additional vertices may be removed (along with their pre-images in Y) without affecting the analysis. Thus, the G-cover $\overline{Y} \to X$ has an unflipped model Y that corresponds to a pair of symmetric multisets S_1, S_2 and an inertia structure $\mathcal{I} = \{I_1, I_2\}$.

In case (ii) the ramification vector is $(3, 2; 0, 0)$, so we must have $I_1 \cong \mathbb{Z}/3\mathbb{Z}$ and $I_2 \cong \mathbb{Z}/2\mathbb{Z}$; choose generators σ, τ of I_1 and I_2 respectively, so that σ has order 3 and τ has order 2 in G. In this case the G-cover $Y \to X$ has no vertical edges, so we may take $S_1 = S_2 = \emptyset$, which implies (since Y is connected) that G is generated by $I_1 \cup I_2$, hence by the generators σ and τ. Defining $\pi : \Gamma \to G$ by $\pi(x) = \tau$ and $\pi(y) = \sigma$ realizes G as a quotient of Γ as required.

Finally, we consider case (iii) with ramification vector $(m_1, m_2; w_1, w_2) = (3, 1; 0, 1)$. Then $I_1 \cong \mathbb{Z}/3\mathbb{Z}$, but I_2 is trivial. As before, choose a generator σ of I_1, which has order 3 in G. Since the fiber over x_1 contains no vertical edges, we may take $S_1 = \emptyset$. The points of the fiber $\overline{Y}_{x_2}$ have vertical multiplicity $v_2 = m_2 w_2 = 1$, which implies that the unflipped model $Y_{x_2} = \mathrm{Cay}(G, \{\tau\})$ for some $\tau \in G$ of order 2. Since Y is connected, it follows that G is generated by $I_1 \cup \{\tau\}$, hence by σ and τ. As in the previous cases, we see that G is a quotient of Γ. This final case is illustrated for $G = \mathfrak{S}_3$ in Figure 6. □

Lawrence University undergraduate Gus Black used Theorem 1.1 to determine the maximal graph groups that arise in low genus:

Genus g	$6(g-1)$	Maximal graph groups for genus g
2	6	$\mathbb{Z}/6\mathbb{Z}, \mathfrak{S}_3$
3	12	$\mathfrak{A}_4$
4	18	$\mathfrak{S}_3 \times \mathbb{Z}/3\mathbb{Z}$
5	24	$\mathfrak{S}_4, \mathfrak{A}_4 \times \mathbb{Z}/2\mathbb{Z}$
6	30	none

TABLE 1. Maximal graph groups for low genus

In particular, $g = 6$ is the first genus for which no maximal graph exists, improving the result established in Proposition 9.2 of [11] that there is no maximal graph of genus 12.

Of course, the modular group $\Gamma \cong PSL_2(\mathbb{Z})$ has been studied intensively due to its central role in number theory and geometry, and much is known about its finite quotients. For instance, a 1901 result of G.A. Miller [17] says that all alternating and symmetric groups are quotients of Γ except for $\mathfrak{A}_6, \mathfrak{A}_7, \mathfrak{A}_8$ and $\mathfrak{S}_5, \mathfrak{S}_6, \mathfrak{S}_8$. Hence, by Theorem 1.1, all alternating and symmetric groups of size at least 6 are maximal graph groups, except for Miller's exceptions. In [14], Liebeck and Shalov

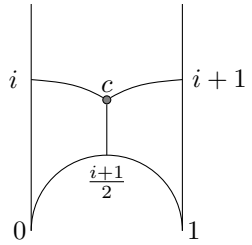

FIGURE 7. The marked ideal triangle T

prove that all but finitely many of the finite simple classical groups different from $PSp_4(2^k)$ and $PSp_4(3^k)$ are quotients of Γ.

4. Connection with Hurwitz surfaces

In [5], Brooks and Makover describe a construction that produces a compact Riemann surface from a 3-regular oriented graph. We briefly summarize their construction here.

Definition 4.1. An *orientation* $\mathcal{O}$ on a 3-regular graph X is the assignment to each vertex $x \in V(X)$ of a cyclic ordering of the edges incident to x. The pair $(X, \mathcal{O})$ is an *oriented* 3-regular graph.

Let T denote the ideal hyperbolic triangle in the upper half-plane with vertices $0, 1$, and ∞. Mark the point $c := \frac{1+i\sqrt{3}}{2}$ inside T, and then draw geodesics from c to the "midpoints" $i, i+1$, and $\frac{i+1}{2}$ of the three sides of T. Hence, we have drawn the half-neighborhood of a vertex in a 3-regular graph on the triangle T. We orient T in the usual clockwise fashion coming from the upper half-plane – this induces the cyclic orientation of the geodesic segments $(i, i+1, \frac{i+1}{2})$.

Starting with an oriented 3-regular graph $(X, \mathcal{O})$, we place one copy of T at each vertex of X, in such a way that the cyclic orientation at c matches the orientation given by $\mathcal{O}$. Each edge of X now connects two copies of T, and we glue these two copies along the corresponding sides by identifying the "midpoints" and matching orientations of the triangles. This procedure results in a complete finite area Riemann surface $\mathcal{S}^O(X, \mathcal{O})$, with cusps corresponding to the vertices of the triangles T. Finally, let $\mathcal{S}(X, \mathcal{O})$ denote the conformal compactification of $\mathcal{S}^O(X, \mathcal{O})$.

Definition 4.2. A *left-hand-turn path* in $(X, \mathcal{O})$ is a closed path in X with the property that at each vertex, the path turns left according to the orientation $\mathcal{O}$. More precisely: a closed path $P = e_1 e_2 \cdots e_n$ is a left-hand-turn path if for all i, the directed edge e_{i+1} follows the directed edge e_i according to the orientation $\mathcal{O}$ (here the index i is taken mod n). The path P is *minimal* if it is not the iteration of a shorter left-hand-turn path. Note that if two minimal left-hand-turn paths share a directed edge, then they are identical.

Proposition 4.3 (see [5] section 4)**.** *Let $(X, \mathcal{O})$ be a 3-regular oriented graph. Then the genus of the corresponding compact Riemann surface $\mathcal{S}(X, \mathcal{O})$ is given by the formula*

$$g(\mathcal{S}(X, \mathcal{O})) = 1 + \frac{|V(X)| - 2L}{4},$$

where L is the number of distinct minimal left-hand-turn paths in $(X, \mathcal{O})$.

Now suppose that G is a maximal graph group. Then by Theorem 1.1, there exists a surjection $\pi : \Gamma \to G$. Fix such a surjection π by choosing generators τ, σ for G of orders 2 and 3 respectively. From the proof of Theorem 1.1, there exists a maximal harmonic G-cover $\overline{Y} \to \star$ with ramification numbers $m = 3$ and $w = 1$. The corresponding unflipped model Y is given by $\mathcal{F}^I(\mathrm{Cay}(G, \{\tau\}))$, where $I = \langle\sigma\rangle$. In particular, $\overline{Y}$ is 3-regular, with vertices labeled by the left cosets of I in G. Moreover, the generator σ cyclically permutes the three edges incident to I, and hence determines a cyclic ordering at the vertex I. The inertia group at a vertex γI is the conjugate subgroup $\gamma I \gamma^{-1}$, with conjugate generator $\gamma\sigma\gamma^{-1}$ that defines a cyclic ordering of the edges incident to γI. Thus, we see that the choice of a surjection π determines a 3-regular oriented graph $(\overline{Y}, \mathcal{O}_\pi)$ on which G acts maximally. Moreover, the G-action preserves the orientation $\mathcal{O}_\pi$. Indeed, pick an edge e incident to I, so that the cyclic order at I is $(e, \sigma e, \sigma^2 e)$. Then an element $\gamma \in G$ sends this triple of edges to the set of edges incident to γI in the order

$$(\gamma e, \gamma\sigma e, \gamma\sigma^2 e) = (\gamma e, (\gamma\sigma\gamma^{-1})\gamma e, (\gamma\sigma\gamma^{-1})^2\gamma e)$$

which matches the orientation at γI. By Proposition 1.7.1 of [16], orientation-preserving automorphisms of an oriented 3-regular graph induce holomorphic automorphisms of the corresponding compact Riemann surface. Hence, from the preceding discussion we see that the harmonic G-action on $\overline{Y}$ yields a (faithful) holomorphic G-action on $\mathcal{S}(\overline{Y}, \mathcal{O}_\pi)$.

Theorem 4.4. *Suppose that G is a maximal graph group, and fix a surjection $\pi : \Gamma \to G$ by choosing generators τ, σ for G of orders 2 and 3 respectively. Let $(\overline{Y}, \mathcal{O}_\pi)$ be the corresponding 3-regular oriented graph, on which G acts maximally. Then we have the following relationship between the order of G, the order of the element $\tau\sigma$ in G, and the genus of the compact Riemann surface $\mathcal{S}(\overline{Y}, \mathcal{O}_\pi)$:*

$$|G|\,(|\tau\sigma| - 6) = 12|\tau\sigma|(g(\mathcal{S}(\overline{Y}, \mathcal{O}_\pi)) - 1).$$

In particular, if G is a Hurwitz group (so $|\tau\sigma| = 7$), this becomes

$$|G| = 84(g(\mathcal{S}(\overline{Y}, \mathcal{O}_\pi)) - 1),$$

so that $\mathcal{S}(\overline{Y}, \mathcal{O}_\pi)$ is a Hurwitz surface.

Proof. By Proposition 4.3, the genus of $\mathcal{S}(\overline{Y}, \mathcal{O}_\pi)$ is given by the following formula, where L is the number of distinct minimal left-hand-turn paths in the oriented graph $(\overline{Y}, \mathcal{O}_\pi)$:

$$\begin{aligned} g(\mathcal{S}(\overline{Y}, \mathcal{O}_\pi)) &= 1 + \frac{|V(\overline{Y})| - 2L}{4} \\ &= 1 + \frac{g(\overline{Y}) - 1 - L}{2} \qquad \text{since } |V(\overline{Y})| = |G/I| = \frac{|G|}{3} = 2g(\overline{Y}) - 2. \end{aligned}$$

A bit of rearrangement yields

$$|G| = 6(g(\overline{Y}) - 1) = 6L + 12(g(\mathcal{S}(\overline{Y}, \mathcal{O}_\pi)) - 1). \tag{4.1}$$

The group G acts on the set of minimal left-hand-turn paths in $\overline{Y}$. In fact, since each directed edge determines a unique minimal left-hand-turn path, and G acts transitively on the set of directed edges, the G-action on the set of minimal left-hand-turn paths is also transitive. Hence, to determine the number L, we just need to compute the order of the stabilizer of a minimal left-hand-turn path. For this, let e be the edge connecting I to τI in $\overline{Y}$, and direct it toward τI. The

orientation at I is given by the cyclic ordering $(e, \sigma e, \sigma^2 e)$, while the orientation at τI is given by $(\tau e, \tau\sigma e, \tau\sigma^2 e) = (e, \tau\sigma e, \tau\sigma^2 e)$, since e is flipped by τ. Thus, a left turn at τI after traveling along e leads to the edge $\tau\sigma e$. It follows that $\tau\sigma$ sends the minimal left-hand-turn path P determined by the directed edge e to itself, so $\langle\tau\sigma\rangle$ is contained in the stabilizer subgroup of P. Moreover, since $\tau\sigma$ shifts the directed edge e onto its successor in P, it follows that the orbit of e under $\langle\tau\sigma\rangle$ contains every directed edge in P. But this implies that $\langle\tau\sigma\rangle$ is the full stabilizer subgroup of P, since otherwise some element of G would fix the directed edge e, contradicting Proposition 2.5. By the Orbit-Stabilizer Theorem, it follows that the number of minimal left-hand-turn paths in $(\overline{Y}, \mathcal{O}_\pi)$ is $L = \frac{|G|}{|\tau\sigma|}$. Putting this into (4.1) and rearranging yields the desired relation:

$$|G|(|\tau\sigma| - 6) = 12|\tau\sigma|(g(\mathcal{S}(\overline{Y})) - 1).$$

□

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