Classification of Groups of Order $n \leq 8$

n=1: The trivial group $\langle e \rangle$ is the only group with 1 element.

n=2,3,5,7: These orders are prime, so Lagrange implies that any such group is cyclic. By the classification of cyclic groups, there is only one group of each order (up to isomorphism):

$$\mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/5\mathbb{Z}, \quad \mathbb{Z}/7\mathbb{Z}.$$

n=4: Here are two groups of order 4:

$$\mathbb{Z}/4\mathbb{Z} \qquad \text{and} \qquad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

(the latter is called the "Klein-four group"). Note that these are not isomorphic, since the first is cyclic, while every non-identity element of the Klein-four has order 2. We will now show that any group of order 4 is either cyclic (hence isomorphic to $\mathbb{Z}/4\mathbb{Z}$) or isomorphic to the Klein-four.

So suppose G is a group of order 4. If G has an element of order 4, then G is cyclic. Hence, we may assume that G has no element of order 4, and try to prove that G is isomorphic to the Klein-four group. Let's give some names to the elements of G:

$$G = \{e, a, b, c\}.$$

Lagrange says that the order of every group element must divide 4, so by our assumption that G is not cyclic, we see that $\text{ord}(a) = \text{ord}(b) = \text{ord}(c) = 2$. I claim that $ab = c$. Indeed, if $ab = e$, then $a = b^{-1}$ contradicting the fact that $b = b^{-1}$. Also, $ab = a$ would imply that $b = e$, while $ab = b$ would imply that $a = e$. The same argument shows that $ba = c = ab$, $ca = b = ac$ and $cb = a = bc$. Using these relations, it is a simple matter (DX) to check that $f : G \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is an isomorphism, where

$$f(e) = (0,0) \quad f(a) = (1,0) \quad f(b) = (0,1) \quad f(c) = (1,1).$$

n=6: You will show for homework that every group of order 6 is isomorphic to one of the following two groups:

$$\mathbb{Z}/6\mathbb{Z} \qquad \text{or} \qquad S_3.$$

n=8: Now let G be a group of order 8. If G has an element of order 8, then G is cyclic, so G is isomorphic to

$$\mathbb{Z}/8\mathbb{Z}.$$

So assume that G has no element of order 8; by Lagrange, every nonidentity element of G has order 2 or 4.

Case 1: Suppose G has no element of order 4, so every nonidentity element has order 2. In other words, every element of G is its own inverse. This means that the function $F : G \to G$ defined by $F(x) = x^{-1}$ is actually the identity function, hence an isomorphism. By a previous homework problem, this implies that G is abelian. Let $a, b \in G$ be two distinct elements of order 2, and choose $c \in G$ distinct from e, a, b, ab. I claim that

$$G = \{e, a, b, c, ab, ac, bc, abc\}.$$

Indeed, these eight elements are distinct by cancelation in G. For instance, if $ab = bc$, then $ab = cb$ since G is abelian. But then cancelation implies that $a = c$, contradicting our choice of c. Now that we know the elements of G, we see (DX) that G can be described via generators and relations as follows:

$$G = \left\langle a, b, c \mid a^2 = b^2 = c^2 = e, ab = ba, ac = ca, bc = cb \right\rangle.$$

Using this description, one can check (DX) that

$$f : G \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

is an isomorphism, where

$$f(a) = (1, 0, 0) \quad f(b) = (0, 1, 0) \quad f(c) = (0, 0, 1).$$

Case 2: Now suppose that $a \in G$ has order 4. Define $H := \langle a \rangle = \{e, a, a^2, a^3\}$, the cyclic subgroup of G generated by a. If b is any element of $G - H$, then $Hb \neq H$, since cosets are either equal or disjoint. Hence, $G = H \cup Hb = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Note that this is true for *any* choice of $b \notin H$. Furthermore, cancelation implies that $ba \neq b$ and $ba \notin \langle a \rangle$ since $b \notin \langle a \rangle$ (DX).

Subcase 2a: Suppose that there exists an element $b \in G - H$ of order 2. Then I claim that $ba \neq a^2b$. For if $ba = a^2b$, then

$$ba^2b = baab = a^2bab = a^2a^2bb = a^4b^2 = ee = e.$$

But then

$$a = ea = b^2a = bba = ba^2b = e,$$

contradicting the fact that a has order 4.

Thus we see that there are two possibilities: either $ba = ab$ or $ba = a^3b$.

Subsubcase 2ai: If $ba = ab$, then G is abelian (DX), and

$$f : G \to \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

is an isomorphism, where

$$f(a) = (1, 0) \quad f(b) = (0, 1).$$

Subsubcase 2aii: If $ba = a^3b$, then I claim that $G \cong D_4$, where D_4 is the group of symmetries of a square. Recall (DX) that

$$D_4 = \left\langle r, f \mid r^4 = e, fr = r^3f \right\rangle,$$

where r stands for a counter-clockwise rotation of $\frac{\pi}{2}$ radians, and f stands for the flip through the diagonal running from the lower left corner to the upper right corner. Using this description, check (DX) that sending a to r and b to f defines an isomorphism

$$G \cong D_4.$$

Subcase 2b: Now suppose that every element of $G - H$ has order 4, and choose one such element b. Then I claim that $b^2 = a^2$. Indeed, since b has order 4, it follows (DX) that b^2 must have order 2, so that $b^2 \in H = \langle a \rangle$. But since a has order 4, the only element of H with order 2 is a^2. Hence $b^2 = a^2$ as claimed.

Now I claim that G is non-abelian. For if $ba = ab$, then

$$(a^3b)^2 = a^3ba^3b = a^6b^2 = a^2b^2 = a^2a^2 = a^4 = e,$$

contradicting the fact that a^3b has order 4. Also, we can't have $ba = a^2b$, for if this were true then

$$ba = a^2b = b^2b = b^3 \implies a = b^2,$$

which is impossible since a has order 4 while b^2 has order 2. Recalling from above that

$$G = H \cup Hb = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\},$$

we see that the only remaining possibility is $ba = a^3b$. We now know enough relations to construct the multiplication table of G, so

$$G = \langle a, b \mid a^4 = b^4 = e, a^2 = b^2, ba = a^3b \rangle =: Q.$$

This is a group that we haven't met before, called the *quaternion group*.

Looking back over our work, we see that up to isomorphism, there are five groups of order 8 (the first three are abelian, the last two non-abelian):

$$\mathbb{Z}/8\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad D_4, \quad Q.$$