Math 300: Problem Set 2

Due Friday, January 22

1. Suppose that $G$ is a group.

   i) (Warm-up) Show that if $(ab)^2 = a^2b^2$ for all $a, b \in G$, then $G$ is abelian.

   ii) Show that if $(ab)^i = a^ib^i$ for three consecutive integers $i$ and all $a, b \in G$, then $G$ is abelian. (Hint: if $n, n+1, n+2$ are the three consecutive integers of the hypothesis, try to show that $ab^n = b^na$ and $ab^{n+1} = b^{n+1}a$ for all $a, b \in G$.)

2. Suppose that $G$ is a nonempty finite set with an associative composition law $G \times G \rightarrow G$ which we denote (as usual) by juxtaposition. Moreover, suppose that two-sided cancelation holds in $G$. This means that if $a, b, c \in G$ and $ab = ac$, then $b = c$, and if $ba = ca$, then $b = c$. In this problem you will prove that $G$ must be a group under this composition law. Follow the outline below:

   i) Choose an element $g \in G$, and consider the function $l_g : G \rightarrow G$ defined by $l_g(x) = gx$ for $x \in G$. Prove that $l_g$ is injective. Why does this imply that $l_g$ is surjective, hence a bijection?

   ii) By i), there exists an element $e \in G$ such that $l_g(e) = g$. Prove that this element $e$ is a left-identity: $ex = x$ for all $x \in G$.

   iii) In this part you will show that $G$ has left-inverses with respect to the element $e$ from ii): for every $g \in G$, there exists an element $g' \in G$ such that $g'g = e$. To do this, choose an arbitrary element $g \in G$, and consider the function $r_g : G \rightarrow G$ defined by $r_g(x) = xg$. Prove that $r_g$ is a bijection, and explain why this shows that $g$ has a left-inverse $g'$.

   iv) You have now shown that $G$ has a left-identity, $e$, and that every $g \in G$ has a left-inverse $g' \in G$ such that $g'g = e$. Show that $e$ is also a right-identity for $G$ (hence an identity), and that $gg' = e$ for all $g \in G$, so that inverses exist as well.

3. Pinter: Chapter 4, Problem E, pp.40-41

4. Pinter: Chapter 4, Problems G1 and G2, p.42

5. Pinter: Chapter 5, Problem A7, p.49

6. Pinter: Chapter 5, Problem D8, p.50
7. In this problem, $G$ is a group.

i) Define the following subset of $G$: $Z(G) := \{g \in G \mid gx = xg \text{ for all } x \in G\}$. Prove that $Z(G)$ is a subgroup of $G$. It is called the center of $G$.

ii) Now let $S \subset G$ be a nonempty subset of $G$. Then define $C_G(S) := \{g \in G \mid gx = xg \text{ for all } x \in S\}$. Prove that $C_G(S)$ is a subgroup of $G$. It is called the centralizer of $S$ in $G$. Note that $Z(G) = C_G(G)$.

iii) If $H = \langle S \rangle$ is the subgroup generated by the subset $S \subset G$, prove that $C_G(H) = C_G(S)$.

8. $G$ is a group, and $H$ is a subgroup of $G$. For each element $g \in G$, consider the set $gHg^{-1} := \{ghg^{-1} \mid h \in H\}$.

i) Prove that for every $g \in G$, the set $gHg^{-1}$ is a subgroup of $G$. It is called the conjugate of $H$ by $g$.

ii) Define $N_G(H) := \{g \in G \mid gHg^{-1} = H\}$. Prove that $N_G(H)$ is a subgroup of $G$. It is called the normalizer of $H$ in $G$.

iii) Prove that $C_G(H)$ is a subgroup of $N_G(H)$.

9. In this problem you will investigate the concepts of centralizer and normalizer in the case of the symmetry group of the triangle, $T$. Recall that in lecture we saw that $T = \langle r, f \mid r^3 = e, f^2 = e, rf = fr^2 \rangle$, where $r$ corresponds to a counterclockwise rotation of $\frac{2\pi}{3}$ radians, and $f$ stands for a flip across the vertical axis of the triangle.

i) List all subgroups of $T$.

ii) Compute the groups $C_T(r), C_T(f), N_T(<r>), N_T(<f>)$.

10. Suppose that $G$ is a group, $a, b \in G$, and $G = \langle a, b \rangle$. That is, $G$ is generated by the elements $a$ and $b$. Suppose further that $a$ and $b$ satisfy the relations $a^2b = ba^3$ and $ba^2 = e$.

What is the order of $G$?