Math 410: Problem Set 1

Due in class Monday, January 9

A) Section 1.2: 7, 19 (the notation in 7 is introduced in Example 3)

B) Section 1.3: 12, 14, 19, 28, 29, 31 (Much of 31 follows immediately from the definition and properties of quotient *groups*. Use anything you want from Math 300, but explain yourself carefully, and be sure to identify what *doesn't* follow automatically from the case of groups.)

C) Section 1.4: 2e, [13 (3rd ed.) OR 15 (4th ed.)]

Problems 28 and 29 from section 1.3 concern the notion of *direct sum*. Before attempting them, read the following paragraphs, and give proofs where indicated.

If V and W are vector spaces over the field $\mathbb{F}$, then the *direct sum* of V and W is defined to be the cartesian product $V \times W$ with component-wise addition and scalar multiplication:

$$(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2) \qquad \text{and} \qquad c \cdot (v, w) := (cv, cw).$$

This construction yields a vector space over $\mathbb{F}$, denoted $V \oplus W$ [Proof?].

Now suppose that W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. Under these conditions, we say that V is the direct sum of W_1 and W_2, because the map $\phi : W_1 \oplus W_2 \to V$ defined by $\phi(w_1, w_2) = w_1 + w_2$ is an isomorphism [Proof?]. (We haven't yet talked about isomorphism in the context of vector spaces, but it means what you think it should from your experience with groups and rings: ϕ is a bijection that preserves addition and scalar multiplication.) Sometimes people distinguish between the spaces $W_1 \oplus W_2$ and V by saying that the former is the *external direct sum* of W_1 and W_2, while V is the *internal direct sum* of W_1 and W_2. This language turns out to be of little value, however, so we won't use it, being content to call either space the direct sum of W_1 and W_2.

Finally, here is another useful characterization of the direct sum: V is the direct sum of two subspaces W_1 and W_2 if and only if every $v \in V$ can be written uniquely as $v = w_1 + w_2$ for some $w_1 \in W_1, w_2 \in W_2$ [Proof?].