Math 410: Problem Set 1

Due in class Monday, January 9

A) Section 1.2: 7, 19 (the notation in 7 is introduced in Example 3)

B) Section 1.3: 12, 14, 19, 28, 29, 31 (Much of 31 follows immediately from the definition and properties of quotient groups. Use anything you want from Math 300, but explain yourself carefully, and be sure to identify what doesn’t follow automatically from the case of groups.)

C) Section 1.4: 2e, [13 (3rd ed.) OR 15 (4th ed.)]

Problems 28 and 29 from section 1.3 concern the notion of direct sum. Before attempting them, read the following paragraphs, and give proofs where indicated.

If $V$ and $W$ are vector spaces over the field $\mathbb{F}$, then the direct sum of $V$ and $W$ is defined to be the cartesian product $V \times W$ with component-wise addition and scalar multiplication:

$$(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c \cdot (v, w) := (cv, cw).$$

This construction yields a vector space over $\mathbb{F}$, denoted $V \oplus W$ [Proof?].

Now suppose that $W_1$ and $W_2$ are subspaces of $V$ such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. Under these conditions, we say that $V$ is the direct sum of $W_1$ and $W_2$, because the map $\phi : W_1 \oplus W_2 \to V$ defined by $\phi(w_1, w_2) = w_1 + w_2$ is an isomorphism [Proof?]. (We haven’t yet talked about isomorphism in the context of vector spaces, but it means what you think it should from your experience with groups and rings: $\phi$ is a bijection that preserves addition and scalar multiplication.) Sometimes people distinguish between the spaces $W_1 \oplus W_2$ and $V$ by saying that the former is the external direct sum of $W_1$ and $W_2$, while $V$ is the internal direct sum of $W_1$ and $W_2$. This language turns out to be of little value, however, so we won’t use it, being content to call either space the direct sum of $W_1$ and $W_2$.

Finally, here is another useful characterization of the direct sum: $V$ is the direct sum of two subspaces $W_1$ and $W_2$ if and only if every $v \in V$ can be written uniquely as $v = w_1 + w_2$ for some $w_1 \in W_1, w_2 \in W_2$ [Proof?].