MIDTERM EXAM #1
APPLIED CALCULUS, MATH 12
WINTER 2001
HUNSICKER

NAME

SOLUTIONS

HONOR PLEDGE

This is a 2 hour exam. Calculators are permitted, but not any other aids. SHOW ALL WORK. If you have used a calculator, indicate how.

DON'T PANIC!!!

I'll be in my office if you have questions.

1) Do the following derivatives (5 points each):

a) Find $f'(-1)$ if $f(x) = 4x^3 - 2x + 1$

$f'(x) = 12x^2 - 2$

$f'(-1) = 12(-1)^2 - 2 = 12 - 2 = 10$

b) Find $f'(t)$ if $f(t) = \frac{1}{t^2} = t^{-2}$

$f'(t) = -2t^{-3} = \frac{-2}{t^3}$

c) Find $f'(t)$ if $f(t) = \sqrt{t} = t^{1/2}$

$f'(t) = \frac{1}{2} t^{-1/2} = \frac{1}{2} \cdot \frac{1}{\sqrt{t}}$
d) Find $f'(1)$ if $f(t) = \frac{1}{(t^2+1)^2} = (t^2+1)^{-2}$

\[ f'(t) = 2(t^2+1)^{-3} \cdot \frac{d}{dt} (t^2+1) \]

\[ = - \frac{2}{(t^2+1)^3} \cdot 2t = - \frac{4t}{(t^2+1)^3} \]

\[ f'(1) = \frac{-4}{(2)^3} = -\frac{1}{2} \]

e) Find $f'(2)$ if $f(t) = \sqrt{t^3+1} = (t^3+1)^{\frac{1}{2}}$

\[ f'(t) = \frac{1}{2} (t^3+1)^{-\frac{1}{2}} \cdot \frac{d}{dt} [t^3+1] \]

\[ = \frac{1}{2\sqrt{t^3+1}} \cdot 3t^2 = \frac{3t^2}{2\sqrt{t^3+1}} \]

\[ f'(2) = \frac{3 \cdot 2^2}{2\sqrt{8+1}} = \frac{3 \cdot 4}{2 \cdot 3} = 2 \]

f) Find $\frac{dV}{dP} \bigg|_{P=2}$ if $V = \frac{1}{2} TP^2$

\[ \frac{dV}{dP} = \frac{1}{2} T \cdot 2P = TP \]

\[ \frac{dV}{dP} \bigg|_{P=2} = 2T \]
2) (10 points) Find $f'(3)$ where $f(x) = x^2 - 1$ using the limits of difference quotients (i.e., the definition of the derivative).

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h)^2 - 1 - 8}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 8}{h}$$

$$= \lim_{h \to 0} \frac{6h + h^2}{h} = \lim_{h \to 0} \frac{6h}{h} = \lim_{h \to 0} 6 = 6$$
3) (15 points) Find the equation of the tangent line to the graph of
\[ f(x) = -x^3 + \frac{1}{x} \]
\[ \text{at the point where } x = 1. \]

\[ f'(x) = -3x^2 - \frac{1}{x^2} \]
\[ f'(1) = -3 - \frac{1}{1^2} = -3 - 1 = -4 \]
\[ f(1) = -(1)^3 + \frac{1}{1} = -1 + 1 = 0 \]

So \[ y - 0 = -4(x - 1) \]
\[ y = -4x + 4 \]
4) The following graph represents the growth of a population of bacteria in a test tube.

a) What was the population on the 5th day? 6000

b) Estimate how fast the population was increasing on the 3rd day.

1000/day

c) Indicate, on the graph, where the population was increasing and decreasing, where faster? Rate, where slower?

d) What was the largest population and when was it reached? What was it increasing fastest? Decreasing fastest?

6000 on day 8, inc. fastest on day 12, dec. fastest just before day 10

e) What days are the 3 day weekend when the grad student forgot to add food to the tube? 8, 9, 10

Explain what might be happening around day 3.

The test tube is reaching its carrying capacity, competition for food is slowing growth.
5) Write a paragraph answer to one of the following questions:

a) Explain how to calculate $f'(2)$ as the limit of slopes of secant lines through the point $(2, f(2))$.

b) In your own words, explain the meaning of $\lim_{x \to 2} x = 3$. Give an example of a function with this property.

To find the slope of a line, we need to know two points on it. We only know one point on the tangent line, so we have to approximate it by slopes of nearby lines for which we know two points. If we take a secant line, that is, a line which intersects the curve $y = f(x)$ at two points, $(2, f(2))$ and $(2+h, f(2+h))$ for some $h$, we can find the slope of that line: \[ \frac{f(2+h) - f(2)}{(2+h) - 2} \] Then this slope will get closer and closer to the tangent slope as $h$ gets smaller and smaller.

So, slope of tangent line $= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$.
\[ \lim_{x \to 2} f(x) = 3 \] means that studying

the values of \( f(x) \) where \( x \) is near 2

leads us to believe \( f(2) \) will be 3.

**Graphic Explanation**

Values of \( f(x) = y \) approach 3 as \( x \) approaches 2.

For this \( x \) value, \( f(x) \approx 3.6 \)

For this \( x \) value, \( f(x) \approx 2.66 \)

More precisely, it means that however close to 3 we want our \( y \) values to be, we can ensure they will be that close by choosing \( x \) close enough to 2.

\[ f(x) = x + 1 \] has this property, as does \( f(x) = \frac{(x+1)(x-2)}{x-1} \).
b) The Nepal earthquake reports that last year, three people were killed in New Year's Eve celebrations by bullets falling back to Earth after having been discharged upwards in celebration. If a bullet is fired from ground level straight up at a velocity of 2400 ft/sec, what is the maximum height it attains? With what velocity does it strike the ground (or an innocent bystander) when it returns to Earth?

\[ x \]

\[ \square \]

\[ \bigcirc \]

constraint equation:
\[ \text{total circumference} = \text{circle} + \text{square} \]
\[ 2\pi r + 4x = 100 \text{ feet} \]

objective equation
\[ \text{total area} = \text{circle} + \text{square} \]
\[ \pi r^2 + x^2 \]

So solve for \( x \) in the constraint equation (or \( r \), it doesn't matter): \[ 4x = 100 - 2\pi r \]
\[ x = 25 - \frac{\pi}{2} r \]

Then plug into objective equation: \[ \pi r^2 + (25 - \frac{\pi}{2} r)^2 = \text{Area}(r) \]
Simplify: \[ \Delta(r) = \pi r^2 + 625 - 25\pi r + \frac{\pi^2}{4} r^2 = \left(\frac{\pi^2}{4} + \pi\right) r^2 - 25\pi r + 625 \]
Take \( \Delta'(r) \) and let it = 0: \[ \Delta'(r) = \left(\frac{\pi^2}{4} + \pi\right) 2r - 25\pi = 0 \] when \[ r = \frac{25\pi}{\left(\frac{\pi^2}{4} + \pi\right) 2} = \frac{25}{\pi^2/4 + \pi} = \frac{50}{\pi + 4} \approx 7 \text{ feet} \]
Will minimize the area... (which will be over 350 square feet)
\[ h(t) = -16t^2 + 2400t + 0 \]

\[ h'(t) = -32t + 2400 \]

The maximum height is attained when \( h'(t) = 0 \), i.e.,

\[ t_{\text{max}} = \frac{2400}{32} = 75 \text{ seconds} \]

\[ h(t) = -16(75)^2 + 2400(75) = 90,000 \text{ feet} \]

It returns to earth when \( h(t) = 0 \),

\[ t_f = \frac{2400}{16} = 150 \text{ seconds} \]

So the velocity then is

\[ h'(150) = -32 \cdot 150 + 2400 = -2400 \text{ feet/second} \]

(Note that if we don't include air resistance, the bullet falls with the same speed as it was discharged. Air resistance is important, but bullets still return to earth with considerable velocity.)