Definitions and Theorems

1) The present value of D dollars to be paid in t years if capital can be invested at rate r is the amount of money that, invested at that rate now, would be worth D dollars in t years.

2) A differential equation is any equation involving an unknown function and its derivatives. An IVP, or initial value problem, is a pair of equations, consisting of one differential equation and one equation which fixes the value of the unknown function at some time.

3) Theorem: \( \ln(x) \) has the properties

   i) \( \ln 1 = 0 \)
   ii) \( \ln(ab) = \ln a + \ln b \)
   iii) \( \ln \left(\frac{a}{b}\right) = \ln a - \ln b \)
   iv) \( \ln \left(\frac{1}{a}\right) = -\ln a \)
   v) \( \ln \left(a^r\right) = r\ln a \)

Essay

4) Come ask me if you have questions about your essay.
Problems

5) a) \[ \begin{align*}
&y' = 0.05y - 50,000 \\
&y(0) = 700,000
\end{align*} \]

is solved by

\[ y(t) = \frac{k}{r} (e^{rt} - 1) + ce^{rt} \]

where \[ \begin{align*}
k &= -50,000 \\
r &= 0.05 \\
 c &= y(0) = 700,000
\end{align*} \]

So \[ y(t) = \frac{-50,000}{0.05} (e^{0.05t} - 1) + 700,000e^{0.05t} \]

\[ = -1,000,000(e^{0.05t} - 1) + 700,000e^{0.05t} \]

\[ = (-1,000,000 + 700,000)e^{0.05t} + 1,000,000 \]

\[ = -300,000e^{0.05t} + 1,000,000 \]

So \[ y(t) = 0 \quad \text{when} \quad 0 = -300,000e^{0.05t} + 1,000,000 \]

\[ 300,000e^{0.05t} = 1,000,000 \]

\[ e^{0.05t} = \frac{1,000,000}{300,000} = \frac{10}{3} \]

\[ 0.05t = \ln \left( \frac{10}{3} \right) \]

\[ t = \frac{\ln \left( \frac{10}{3} \right)}{0.05} \approx 24 \]

b) \[ \frac{d}{dx} [x^{2x}] = x^{2x} \frac{d}{dx} [\ln (x^{2x})] = x^{2x} \frac{d}{dx} [2x \ln x] \]

\[ = x^{2x} \left( 2 \ln x + 2x \cdot \frac{1}{x} \right) = x^{2x} \left[ 2 \ln x + 2 \right] \]
(6) The APYs for the two accounts are:

\[
\text{APY}_1 = \left(1 + \frac{0.051}{4}\right)^4 - 1 \cong 0.0519 \implies 5.19% \\
\text{APY}_2 = \left(1 + \frac{0.05}{365}\right)^{365} - 1 \cong 0.0512 \implies 5.12% 
\]

So the first account gives more interest.

(7) a) \( y' = 0.2y - 500,000 = 0 \) when

\[
0.2y = 500,000 \\
y = 2,500,000 \quad \text{is the eq. solution}
\]

\( y' = 0.2y - \frac{y}{100} = 0 \) when

\[
0.2y = \frac{y}{100} \\
y = \frac{100y}{20} = 5y \\
y = 5f \quad \text{is the eq. solution}
\]

If the initial value for \( y \) is the equilibrium value, then \( y \) remains constant over time, re, the population does not change.

b) 

If \( y = 10f \), then \( y' = 2f - f = f > 0 \)

so \( y \) is increasing.

If \( y = 0 \), then \( y' = -f < 0 \),

so \( y \) is decreasing.
c) \( y = 5f \) is the equilibrium solution, this equals 7,000,000 ft

\[
5f = 7,000,000 \\
f = \frac{1}{5} \cdot 7,000,000 = 1,400,000 \text{ ft}\text{/year.}
\]

8) a) \( f'(t) = r [f(t)]^2 \) 

\( r < 0 \), since the more water is in the tank, the faster it seeps out, i.e., the faster the amount decreases.

b) \( f'(t) = r [f(t)]^2 + p \) 

\( p \) is positive, since this water pumped in increases the amount in the tank.