Hw 4
Selected Solutions

13.3 27) Where does $y=e^x$ have maximal curvature? For the curve $y=f(x)$,

$$\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \frac{e^x}{(1+e^{2x})^{3/2}}$$

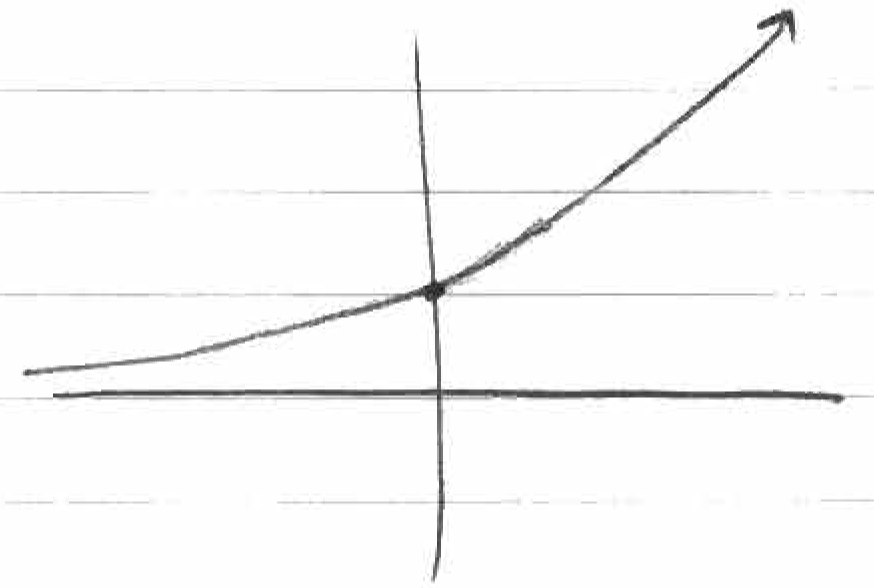
To find max, take κ' and set $=0$:

$$\kappa'(x) = \frac{e^x(1+e^{2x})^{3/2} - 2(1+e^{2x})^{1/2}\cdot e^x}{(1+e^{2x})^3} = 0$$

When the numerator $=0$:

$$e^x(1+e^{2x})^{3/2} - 2e^x(1+e^{2x})^{1/2} = 0$$
$$e^x(1+e^{2x})^{1/2}\left[1+e^{2x}-2\right] = 0$$
$$e^{2x} = 1$$
$$x = 0$$

So the maximal curvature is at $(0,1)$.

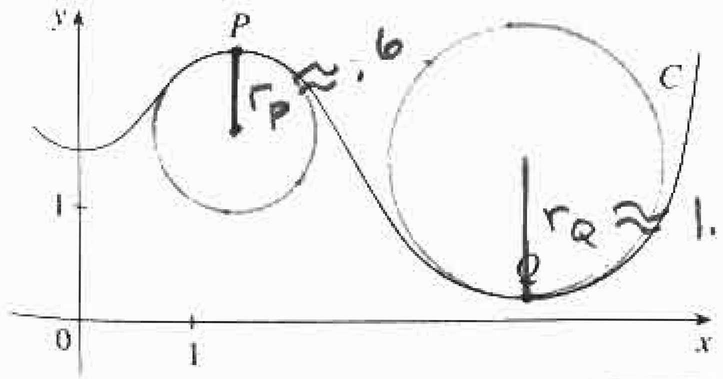


as $x \to \infty$, $\dfrac{e^x}{(1+e^{2x})^{3/2}} \approx \dfrac{e^x}{e^{3x}} \approx \dfrac{1}{e^{2x}} \longrightarrow 0$

29)

29. (a) Is the curvature of the curve C shown in the figure greater at P or at Q? Explain.

(b) Estimate the curvature at P and at Q by sketching the osculating circles at those points.



a) $K = \dfrac{1}{\text{radius of osc. circle}}$

so since $r_P < r_Q$, $K_Q < K_P$

b) $K_P \approx \frac{10}{6}$ $K_Q \approx \frac{10}{13}$

39) Find $\vec{T}, \vec{N}, \vec{B}$ for $\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$ at $(1, \frac{2}{3}, 1)$

Note $t=1$ here,

so we want $\vec{T}(1), \vec{N}(1), \vec{B}(1)$.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle 2t, 2t^2, 1 \rangle}{\sqrt{4t^2+4t^4+1}} = \frac{\langle 2t, 2t^2, 1 \rangle}{\sqrt{(2t^2+1)^2}} = \frac{1}{(2t^2+1)} \langle 2t, 2t^2, 1 \rangle$$

$$\vec{T}(1) = \langle \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3} \rangle$$

$$T'(t) = \left(\frac{1}{(2t^2+1)} \langle 2t, 2t^2, 1 \rangle \right)' = \frac{-4t}{(2t^2+1)^2} \langle 2t, 2t^2, 1 \rangle + \frac{1}{(2t^2+1)} \langle 2, 4t, 0 \rangle$$

$$= \frac{1}{(2t^2+1)^2} \left[\langle -8t^2, -8t^3, -4t \rangle + \langle 4t^2+2, 8t^3+4t, 0 \rangle \right]$$

$$= \frac{1}{(2t^2+1)^2} \langle -4t^2+2, 4t, -4t \rangle$$

so $\vec{T}\,'(1) = \frac{1}{9} \langle -2, 4, -4 \rangle = \frac{2}{9} \langle -1, 2, -2 \rangle$

so $\vec{N}(1) = \dfrac{\vec{T}\,'(1)}{|\vec{T}\,'(1)|} = \dfrac{\langle -1, 2, -2 \rangle}{|\langle -1, 2, -2 \rangle|} = \langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle$

and $\vec{B}(1) = \vec{T}(1) \times \vec{N}(1) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle \times \langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle = \langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle$.

41) Find the equations of the normal and osculating planes of $\vec{r}(t) = \langle 2\sin 3t, t, 2\cos 3t \rangle$ at $(0, \pi, -2)$

Here $t = \pi$, so we need $\vec{T}(\pi), \vec{N}(\pi), \vec{B}(\pi)$:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle 6\cos 3t, 1, -6\sin 3t \rangle}{\sqrt{36\cos^2 3t + 1 + 36\sin^2 3t}} = \frac{1}{\sqrt{37}} \langle 6\cos 3t, 1, -6\sin 3t \rangle$$

$$\vec{T}\,'(t) = \frac{1}{\sqrt{37}} \langle -18\sin 3t, 0, -18\cos 3t \rangle = \frac{18}{\sqrt{37}} \langle -\sin 3t, 0, -\cos 3t \rangle$$

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|T'(t)|} = \langle -\sin 3t, 0, -\cos 3t \rangle$$ since this is a unit vector

so $\vec{T}(\pi) = \frac{1}{\sqrt{37}} \langle -6, 1, 0 \rangle \quad \vec{N}(\pi) = \langle 0, 0, 1 \rangle$

so $\vec{B}(\pi) = \vec{T}(\pi) \times \vec{N}(\pi) = \frac{1}{\sqrt{37}} \left(\langle -6, 1, 0 \rangle \times \langle 0, 0, 1 \rangle \right)$

$$= \frac{1}{\sqrt{37}} \begin{vmatrix} i & j & k \\ -6 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{37}} \left(i \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - j \begin{vmatrix} -6 & 0 \\ 0 & 1 \end{vmatrix} + k \begin{vmatrix} -6 & 1 \\ 0 & 0 \end{vmatrix} \right)$$

$$= \frac{1}{\sqrt{37}} \langle 1, 6, 0 \rangle$$

<u>Normal plane</u>: contains $(0, \pi, -2)$ and $\perp \vec{T}$, ie $\perp \langle -6, 1, 0 \rangle$

$-6x + y = \langle -6, 1, 0 \rangle \cdot \langle 0, \pi, -2 \rangle = \pi$

$-6x + y = \pi$

<u>Osculating</u>: contains $(0, \pi, -2)$ and $\perp \vec{B}$, ie $\perp \langle 1, 6, 0 \rangle$

$x + 6y = \langle 0, \pi, -2 \rangle \cdot \langle 1, 6, 0 \rangle$

$x + 6y = 6\pi$