53. Note that here the variables are m and b, and $f(m,b) = \sum_{i=1}^{n} [y_i - (mx_i + b)]^2$. Then

$f_m = \sum_{i=1}^{n} -2x_i[y_i - (mx_i + b)] = 0$ implies $\sum_{i=1}^{n} (x_i y_i - m x_i^2 - b x_i) = 0$ or $\sum_{i=1}^{n} x_i y_i = m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i$

and $f_b = \sum_{i=1}^{n} -2[y_i - (mx_i + b)] = 0$ implies $\sum_{i=1}^{n} y_i = m \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} b = m\left(\sum_{i=1}^{n} x_i\right) + nb$. Thus we have

the two desired equations. Now $f_{mm} = \sum_{i=1}^{n} 2x_i^2$, $f_{bb} = \sum_{i=1}^{n} 2 = 2n$ and $f_{mb} = \sum_{i=1}^{n} 2x_i$. And $f_{mm}(m,b) > 0$

always and $D(m,b) = 4n\left(\sum_{i=1}^{n} x_i^2\right) - 4\left(\sum_{i=1}^{n} x_i\right)^2 = 4\left[n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2\right] > 0$ always so the

solutions of these two equations do indeed minimize $\sum_{i=1}^{n} d_i^2$.

14.8

3. $f(x,y) = x^2 - y^2$, $g(x,y) = x^2 + y^2 = 1 \Rightarrow \nabla f = \langle 2x, -2y \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2x = 2\lambda x$ implies $x = 0$ or $\lambda = 1$. If $x = 0$, then $x^2 + y^2 = 1$ implies $y = \pm 1$ and if $\lambda = 1$, then $-2y = 2\lambda y$ implies $y = 0$ and thus $x = \pm 1$. Thus the possible points for the extreme values of f are $(\pm 1, 0)$, $(0, \pm 1)$. But $f(\pm 1, 0) = 1$ while $f(0, \pm 1) = -1$ so the maximum value of f on $x^2 + y^2 = 1$ is $f(\pm 1, 0) = 1$ and the minimum value is $f(0, \pm 1) = -1$.

7. $f(x,y,z) = 2x + 6y + 10z$, $g(x,y,z) = x^2 + y^2 + z^2 = 35 \Rightarrow \nabla f = \langle 2, 6, 10 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$. Then $2\lambda x = 2$, $2\lambda y = 6$, $2\lambda z = 10$ imply $x = \frac{1}{\lambda}$, $y = \frac{3}{\lambda}$, and $z = \frac{5}{\lambda}$. But $35 = x^2 + y^2 + z^2 = \left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 + \left(\frac{5}{\lambda}\right)^2 \Rightarrow 35 = \frac{35}{\lambda^2} \Rightarrow \lambda = \pm 1$, so f has possible extreme values at the points $(1,3,5)$, $(-1,-3,-5)$. The maximum value of f on $x^2 + y^2 + z^2 = 35$ is $f(1,3,5) = 70$, and the minimum is $f(-1,-3,-5) = -70$.

11. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x^4 + y^4 + z^4 = 1 \Rightarrow$ $\nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \langle 4\lambda x^3, 4\lambda y^3, 4\lambda z^3 \rangle$.
Case 1: If $x \neq 0$, $y \neq 0$ and $z \neq 0$, then $\nabla f = \lambda \nabla g$ implies $\lambda = 1/(2x^2) = 1/(2y^2) = 1/(2z^2)$ or $x^2 = y^2 = z^2$ and $3x^4 = 1$ or $x = \pm \frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$ all with an f-value of $\sqrt{3}$.
Case 2: If one of the variables equals zero and the other two are not zero, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{\sqrt{2}}$ and corresponding f value of $\sqrt{2}$.
Case 3: If exactly two of the variables are zero, then the third variable has value ± 1 with the corresponding f value of 1. Thus on $x^4 + y^4 + z^4 = 1$, the maximum value of f is $\sqrt{3}$ and the minimum value is 1.

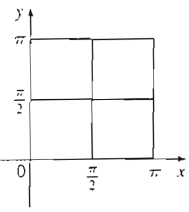
15. $f(x,y,z) = x + 2y$, $g(x,y,z) = x + y + z = 1$, $h(x,y,z) = y^2 + z^2 = 4 \Rightarrow \nabla f = \langle 1, 2, 0 \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$ and $\mu \nabla h = \langle 0, 2\mu y, 2\mu z \rangle$. Then $1 = \lambda$, $2 = \lambda + 2\mu y$ and $0 = \lambda + 2\mu z$ so $\mu y = \frac{1}{2} = -\mu z$ or $y = 1/(2\mu)$, $z = -1/(2\mu)$. Thus $x + y + z = 1$ implies $x = 1$ and $y^2 + z^2 = 4$ implies $\mu = \pm \frac{1}{2\sqrt{2}}$. Then the possible points are $\left(1, \pm\sqrt{2}, \mp\sqrt{2}\right)$ and the maximum value is $f\left(1, \sqrt{2}, -\sqrt{2}\right) = 1 + 2\sqrt{2}$ and the minimum value is $f\left(1, -\sqrt{2}, \sqrt{2}\right) = 1 - 2\sqrt{2}$.

21. $P(L,K) = bL^\alpha K^{1-\alpha}$, $g(L,K) = mL + nK = p \Rightarrow \nabla P = \left\langle \alpha b L^{\alpha-1} K^{1-\alpha}, (1-\alpha) b L^\alpha K^{-\alpha} \right\rangle$, $\lambda \nabla g = \langle \lambda m, \lambda n \rangle$. Then $\alpha b (K/L)^{1-\alpha} = \lambda m$ and $(1-\alpha) b (L/K)^\alpha = \lambda n$ and $mL + nK = p$, so $\alpha b (K/L)^{1-\alpha}/m = (1-\alpha) b (L/K)^\alpha / n$ or $n\alpha/[m(1-\alpha)] = (L/K)^\alpha (L/K)^{1-\alpha}$ or $L = Kn\alpha/[m(1-\alpha)]$. Substituting into $mL + nK = p$ gives $K = (1-\alpha)p/n$ and $L = \alpha p/m$ for the maximum production.

35. $f(x,y,z) = xyz$, $g(x,y,z) = 4(x+y+z) = c \Rightarrow \nabla f = \langle yz, xz, xy \rangle$, $\lambda \nabla g = \langle 4\lambda, 4\lambda, 4\lambda \rangle$. Thus $4\lambda = yz = xz = xy$ or $x = y = z = \frac{1}{12}c$ are the dimensions giving the maximum volume.

15.1

3. (a) The subrectangles are shown in the figure. Since $\Delta A = \pi^2/4$, we estimate



$$\iint_R \sin(x+y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$
$$= f(0,0)\,\Delta A + f(0,\tfrac{\pi}{2})\,\Delta A + f(\tfrac{\pi}{2},0)\,\Delta A + f(\tfrac{\pi}{2},\tfrac{\pi}{2})\,\Delta A$$
$$= 0\left(\tfrac{\pi^2}{4}\right) + 1\left(\tfrac{\pi^2}{4}\right) + 1\left(\tfrac{\pi^2}{4}\right) + 0\left(\tfrac{\pi^2}{4}\right) = \tfrac{\pi^2}{2} \approx 4.935$$

9. (a) With $m = n = 2$, we have $\Delta A = 4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\,\Delta A = \Delta A[f(1,1) + f(1,3) + f(3,1) + f(3.3)]$$
$$\approx 4(27 + 4 + 14 + 17) = 248$$

(b) $f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA \approx \frac{1}{16}(248) = 15.5$

11. $z = 3 > 0$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 3$ and above the rectangle $[-2,2] \times [1,6]$. S is a rectangular solid, thus $\iint_R 3\,dA = 4 \cdot 5 \cdot 3 = 60$.

15.2

7. $\int_0^2 \int_0^1 (2x+y)^8 \, dx\, dy = \int_0^2 \left[\frac{1}{2} \frac{(2x+y)^9}{9} \right]_{x=0}^{x=1} dy$ [substitute $u = 2x + y \Rightarrow dx = \frac{1}{2} du$]

$$= \frac{1}{18} \int_0^2 [(2+y)^9 - (0+y)^9] \, dy = \frac{1}{18} \left[\frac{(2+y)^{10}}{10} - \frac{y^{10}}{10} \right]_0^2$$

$$= \frac{1}{180} [(4^{10} - 2^{10}) - (2^{10} - 0^{10})] = \frac{1{,}046{,}528}{180} = \frac{261{,}632}{45}$$

11. $\int_0^{\ln 2} \int_0^{\ln 5} e^{2x-y} \, dx\, dy = \left(\int_0^{\ln 5} e^{2x} \, dx \right) \left(\int_0^{\ln 2} e^{-y} \, dy \right) = \left[\frac{1}{2} e^{2x} \right]_0^{\ln 5} \left[-e^{-y} \right]_0^{\ln 2}$

$$= \left(\tfrac{25}{2} - \tfrac{1}{2} \right) \left(-\tfrac{1}{2} + 1 \right) = 6$$

15. $\iint_R \frac{xy^2}{x^2+1} \, dA = \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1} \, dy\, dx = \int_0^1 \frac{x}{x^2+1} \, dx \int_{-3}^3 y^2 \, dy$

$$= \left[\tfrac{1}{2} \ln(x^2+1) \right]_0^1 \left[\tfrac{1}{3} y^3 \right]_{-3}^3 = \tfrac{1}{2} (\ln 2 - \ln 1) \cdot \tfrac{1}{3} (27 + 27) = 9 \ln 2$$

19. $\iint_R xye^{x^2 y} \, dA = \int_0^2 \int_0^1 xye^{x^2 y} \, dx\, dy = \int_0^2 \left[\frac{1}{2} e^{x^2 y} \right]_{x=0}^{x=1} dy = \frac{1}{2} \int_0^2 (e^y - 1) \, dy$

$$= \tfrac{1}{2} \left[e^y - y \right]_0^2 = \tfrac{1}{2} [(e^2 - 2) - (1 - 0)] = \tfrac{1}{2} (e^2 - 3)$$

36. (a) Loosely speaking, Fubini's Theorem says that the order of integration of a function of two variables does not affect the value of the double integral, while Clairaut's Theorem says that the order of differentiation of such a function does not affect the value of the second-order derivative. Also, both theorems require continuity (though Fubini's allows a finite number of smooth curves to contain discontinuities).

(b) To find g_{xy}, we first hold y constant and use the single-variable Fundamental Theorem of Calculus, Part 1: $g_x = \frac{d}{dx} g(x, y) = \frac{d}{dx} \int_a^x \left(\int_c^y f(s, t) \, dt \right) ds = \int_c^y f(x, t) \, dt$. Now we use the Fundamental Theorem again: $g_{xy} = \frac{d}{dy} \int_c^y f(x, t) \, dt = f(x, y)$.

To find g_{yx}, we first use Fubini's Theorem to find that $\int_a^x \int_c^y f(s, t) \, dt\, ds = \int_c^y \int_a^x f(s, t) \, dt\, ds$, and then use the Fundamental Theorem twice, as above, to get $g_{yx} = f(x, y)$. So $g_{xy} = g_{yx} = f(x, y)$.