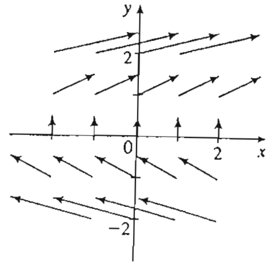
Hw #9

16.1

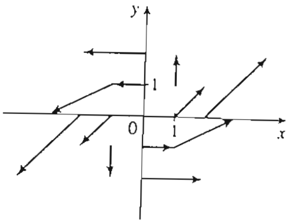
3. $\mathbf{F}(x, y) = y\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$.

The length of the vector $y\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$ is $\sqrt{y^2 + \frac{1}{4}}$. Vectors are tangent to parabolas opening about the x-axis.



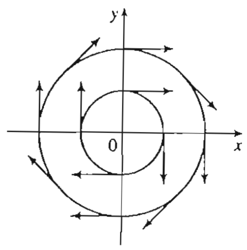
4. $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + x\,\mathbf{j}$

The length of the vector $(x - y)\,\mathbf{i} + x\,\mathbf{j}$ is $\sqrt{(x-y)^2 + x^2}$. Vectors along the line $y = x$ are vertical.



6. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

All the vectors $\mathbf{F}(x, y)$ are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2 + y^2}$.



11. $\mathbf{F}(x, y) = \langle y, x\rangle$ corresponds to graph II. In the first quadrant all the vectors have positive x- and y-components, in the second quadrant all vectors have positive x-components and negative y-components, in the third quadrant all vectors have negative x- and y-components, and in the fourth quadrant all vectors have negative x-components and positive y-components. In addition, the vectors get shorter as we approach the origin.

12. $\mathbf{F}(x, y) = \langle 1, \sin y \rangle$ corresponds to graph IV since the x-component of each vector is constant, the vectors are independent of x (vectors along horizontal lines are identical), and the vector field appears to repeat the same pattern vertically.

13. $\mathbf{F}(x, y) = \langle x - 2, x + 1 \rangle$ corresponds to graph I since the vectors are independent of y (vectors along vertical lines are identical) and, as we move to the right, both the x- and the y-components get larger.

14. $\mathbf{F}(x, y) = \langle y, 1/x \rangle$ corresponds to graph III. As in Exercise 11, all the vectors in the first quadrant have positive x- and y-components, in the second quadrant all vectors have positive x-components and negative y-components, in the third quadrant all vectors have negative x- and y-components, and in the fourth quadrant all vectors have negative x-components and positive y-components. Also, the vectors become longer as we approach the y-axis.

15. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ corresponds to graph IV, since all vectors have identical length and direction.

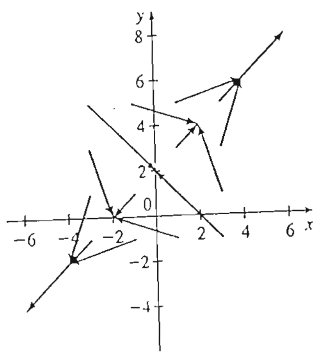
16. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + z\mathbf{k}$ corresponds to graph I, since the horizontal vector components remain constant, but the vectors above the xy-plane point generally upward while the vectors below the xy-plane point generally downward.

17. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph III; the projection of each vector onto the xy-plane is $x\,\mathbf{i} + y\,\mathbf{j}$, which points away from the origin, and the vectors point generally upward because their z-components are all 3.

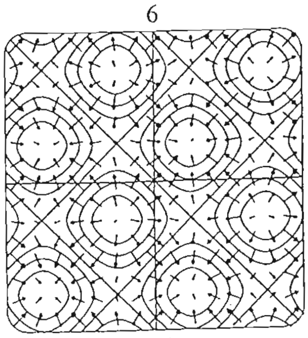
18. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ corresponds to graph II; each vector $\mathbf{F}(x, y, z)$ has the same length and direction as the position vector of the point (x, y, z), and therefore the vectors all point directly away from the origin.

21. $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \dfrac{1}{x + 2y}\,\mathbf{i} + \dfrac{2}{x + 2y}\,\mathbf{j}$

25. $f(x, y) = xy - 2x \quad \Rightarrow$
$\nabla f(x, y) = (y - 2)\,\mathbf{i} + x\,\mathbf{j}$.
The length of $\nabla f(x, y)$ is $\sqrt{(y - 2)^2 + x^2}$ and $\nabla f(x, y)$ terminates on the line $y = x + 2$ at the point $(x + y - 2, x + y)$.



27. We graph ∇f along with a contour map of f.



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

29. $f(x, y) = xy \quad \Rightarrow \quad \nabla f(x, y) = y\,\mathbf{i} + x\,\mathbf{j}$. In the first quadrant, both components of each vector are positive, while in the third quadrant both components are negative. However, in the second quadrant each vector's x-component is positive while its y-component is negative (and vice versa in the fourth quadrant). Thus, ∇f is graph IV.

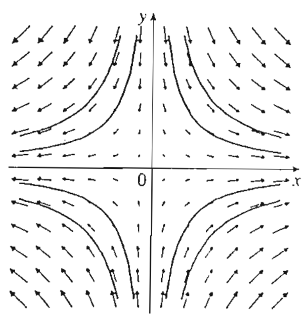
30. $f(x, y) = x^2 - y^2 \quad \Rightarrow \quad \nabla f(x, y) = 2x\,\mathbf{i} - 2y\,\mathbf{j}$. In the first quadrant, the x-component of each vector is positive while the y-component is negative. The other three quadrants are similar, where the x-component of each vector has the same sign as the x-value of its initial point, and the y-component has sign opposite that of the y-value of the initial point. Thus, ∇f is graph III.

31. $f(x, y) = x^2 + y^2 \quad \Rightarrow \quad \nabla f(x, y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Thus, each vector $\nabla f(x, y)$ has the same direction and twice the length of the position vector of the point (x, y), so the vectors all point directly away from the origin and their lengths increase as we move away from the origin. Hence, ∇f is graph II.

32. $f(x, y) = \sqrt{x^2 + y^2} \quad \Rightarrow \quad \nabla f(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j}$. Then

$|\nabla f(x, y)| = \dfrac{1}{\sqrt{x^2 + y^2}}\sqrt{x^2 + y^2} = 1$, so all vectors are unit vectors. In addition, each vector $\nabla f(x, y)$ has the same direction as the position vector of the point (x, y), so the vectors all point directly away from the origin. Hence, ∇f is graph I.

33. (a) We sketch the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be hyperbolas with shape similar to the graph of $y = \pm 1/x$, so we might guess that the flow lines have equations $y = C/x$.



16.2

1. $x = t^2$ and $y = t$, $0 \le t \le 2$, so by Formula 3

$$\int_C y\,ds = \int_0^2 t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^2 t\sqrt{(2t)^2 + (1)^2}\,dt$$
$$= \int_0^2 t\sqrt{4t^2+1}\,dt = \tfrac{1}{12}\left(4t^2+1\right)^{3/2}\Big]_0^2 = \tfrac{1}{12}\left(17\sqrt{17}-1\right)$$

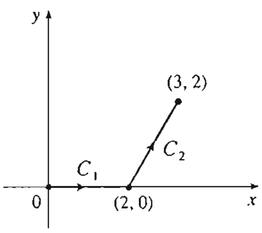
3. Parametric equations for C are $x = 4\cos t$, $y = 4\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2}(4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt$$
$$= \int_{-\pi/2}^{\pi/2} 4^5 \cos t \sin^4 t\sqrt{16\left(\sin^2 t + \cos^2 t\right)}\,dt$$
$$= 4^5\int_{-\pi/2}^{\pi/2}\left(\sin^4 t\cos t\right)(4)\,dt = (4)^6\left[\tfrac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = \frac{2\cdot 4^6}{5} = 1638.4$$

5. If we choose x as the parameter, parametric equations for C are $x = x$, $y = x^2$ for $1 \le x \le 3$ and

$$\int_C (xy + \ln x)\,dy = \int_1^3\left(x\cdot x^2 + \ln x\right)2x\,dx = \int_1^3 2\left(x^4 + x\ln x\right)dx$$
$$= 2\left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2\right]_1^3 \quad \text{(by integrating by parts in the second term)}$$
$$= 2\left(\tfrac{243}{5} + \tfrac{9}{2}\ln 3 - \tfrac{9}{4} - \tfrac{1}{5} + \tfrac{1}{4}\right) = \tfrac{464}{5} + 9\ln 3$$

7.



$C = C_1 + C_2$

On C_1: $x = x$, $y = 0 \;\Rightarrow\; dy = 0\,dx$, $0 \le x \le 2$.

On C_2: $x = x$, $y = 2x - 4 \;\Rightarrow\; dy = 2\,dx$, $2 \le x \le 3$.

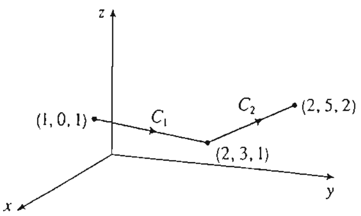
Then

$$\int_C xy\,dx + (x-y)\,dy = \int_{C_1} xy\,dx + (x-y)\,dy + \int_{C_2} xy\,dx + (x-y)\,dy$$
$$= \int_0^2 (0+0)\,dx + \int_2^3\left[\left(2x^2 - 4x\right) + (-x+4)(2)\right]dx$$
$$= \int_2^3\left(2x^2 - 6x + 8\right)dx = \tfrac{17}{3}$$

11. Parametric equations for C are $x = t$, $y = 2t$, $z = 3t$, $0 \le t \le 1$. Then

$$\int_C xe^{yz}\,ds = \int_0^1 te^{(2t)(3t)}\sqrt{1^2+2^2+3^2}\,dt = \sqrt{14}\int_0^1 te^{6t^2}\,dt$$
$$= \sqrt{14}\left[\tfrac{1}{12}e^{6t^2}\right]_0^1 = \tfrac{\sqrt{14}}{12}\left(e^6 - 1\right)$$

15.



On C_1: $x = 1 + t \;\Rightarrow\; dx = dt$, $y = 3t \;\Rightarrow\; dy = 3\,dt$, $z = 1$ $\;\Rightarrow\; dz = 0\,dt$, $0 \le t \le 1$.

On C_2: $x = 2 \;\Rightarrow\; dx = 0\,dt$, $y = 3 + 2t \;\Rightarrow\; dy = 2\,dt$, $z = 1 + t \;\Rightarrow\; dz = dt$, $0 \le t \le 1$.

Then $\int_C (x + yz)\,dx + 2x\,dy + xyz\,dz$

$$= \int_{C_1}(x+yz)\,dx + 2x\,dy + xyz\,dz + \int_{C_2}(x+yz)\,dx + 2x\,dy + xyz\,dz$$
$$= \int_0^1 (1 + t + (3t)(1))\,dt + 2(1+t)\cdot 3\,dt + (1+t)(3t)(1)\cdot 0\,dt$$
$$+ \int_0^1 (2 + (3+2t)(1+t))\cdot 0\,dt + 2(2)\cdot 2\,dt + (2)(3+2t)(1+t)dt$$
$$= \int_0^1 (10t + 7)\,dt + \int_0^1\left(4t^2 + 10t + 14\right)dt$$
$$= \left[5t^2 + 7t\right]_0^1 + \left[\tfrac{4}{3}t^3 + 5t^2 + 14t\right]_0^1 = 12 + \tfrac{61}{3} = \tfrac{97}{3}$$

17. (a) Along the line $x = -3$, the vectors of $\mathbf{F}$ have positive y-components, so since the path goes upward, the integrand $\mathbf{F} \cdot \mathbf{T}$ is always positive. Therefore $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds$ is positive.

(b) All of the (nonzero) field vectors along the circle with radius 3 are pointed in the clockwise direction, that is, opposite the direction to the path. So $\mathbf{F} \cdot \mathbf{T}$ is negative, and therefore $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds$ is negative.

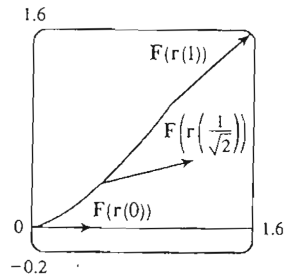
19. $\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\mathbf{j}$, so $\mathbf{F}(\mathbf{r}(t)) = (t^2)^2(-t^3)^3\,\mathbf{i} - (-t^3)\sqrt{t^2}\,\mathbf{j} = -t^{13}\,\mathbf{i} + t^4\,\mathbf{j}$ and $\mathbf{r}'(t) = 2t\,\mathbf{i} - 3t^2\,\mathbf{j}$.
Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 \left(-2t^{14} - 3t^6\right) dt = \left[-\frac{2}{15}t^{15} - \frac{3}{7}t^7\right]_0^1 = -\frac{59}{105}$.

25. (a) $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left\langle e^{t^2-1}, t^5 \right\rangle \cdot \langle 2t, 3t^2 \rangle\, dt = \int_0^1 \left(2te^{t^2-1} + 3t^7\right) dt = \left[e^{t^2-1} + \frac{3}{8}t^8\right]_0^1 = \frac{11}{8} - 1/e$

(b) $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{F}(\mathbf{r}(0)) = \langle e^{-1}, 0 \rangle$;

$\mathbf{r}\left(\frac{1}{\sqrt{2}}\right) = \left\langle \frac{1}{2}, \frac{1}{2\sqrt{2}} \right\rangle$, $\mathbf{F}\left(\mathbf{r}\left(\frac{1}{\sqrt{2}}\right)\right) = \left\langle e^{-1/2}, \frac{1}{4\sqrt{2}} \right\rangle$;

$\mathbf{r}(1) = \langle 1, 1 \rangle$, $\mathbf{F}(\mathbf{r}(1)) = \langle 1, 1 \rangle$.



In order to generate the graph with Maple, we use the `PLOT` command (not to be confused with the `plot` command) to define each of the vectors. For example,

```
v1:=PLOT(CURVES([[0,0],[evalf(1/exp(1)),0]]));
```

generates the vector from the vector field at the point $(0, 0)$ (but without an arrowhead) and gives it the name `v1`. To show everything on the same screen, we use the `display` command. In Mathematica, we use `ListPlot` (with the `PlotJoined -> True` option) to generate the vectors, and then `Show` to show everything on the same screen.

$$W = \int_{-1}^{2} \langle x \sin x^2, x^2 \rangle \cdot \langle 1, 2x \rangle\, dx = \int_{-1}^{2} (x \sin x^2 + 2x^3)\, dx = \left[-\tfrac{1}{2}\cos x^2 + \tfrac{1}{2}x^4\right]_{-1}^{2}$$
$$= \tfrac{1}{2}(15 + \cos 1 - \cos 4)$$

39. $\mathbf{r}(t) = \langle 1 + 2t, 4t, 2t \rangle$, $0 \le t \le 1$,

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 6t, 1 + 4t, 1 + 6t \rangle \cdot \langle 2, 4, 2 \rangle\, dt = \int_0^1 (12t + 4(1 + 4t) + 2(1 + 6t))\, dt$$
$$= \int_0^1 (40t + 6)\, dt = \left[20t^2 + 6t\right]_0^1 = 26$$

46. Use the orientation pictured in the figure. Then since $\mathbf{B}$ is tangent to any circle that lies in the plane perpendicular to the wire, $\mathbf{B} = |\mathbf{B}|\,\mathbf{T}$ where $\mathbf{T}$ is the unit tangent to the circle C: $x = r\cos\theta$, $y = r\sin\theta$. Thus
$\mathbf{B} = |\mathbf{B}|\,\langle -\sin\theta, \cos\theta \rangle$. Then
$\int_C \mathbf{B} \cdot d\mathbf{r} = \int_0^{2\pi} |\mathbf{B}|\,\langle -\sin\theta, \cos\theta \rangle \cdot \langle -r\sin\theta, r\cos\theta \rangle\, d\theta = \int_0^{2\pi} |\mathbf{B}|\, r\, d\theta = 2\pi r\,|\mathbf{B}|$. (Note that $|\mathbf{B}|$ here is the magnitude of the field at a distance r from the wire's center.) But by Ampere's Law $\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$. Hence $|\mathbf{B}| = \mu_0 I/(2\pi r)$.