A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

\( \text{(S) Surface area} = 4\pi r^2 \)

\( \text{(V) Volume} = \frac{4}{3} \pi r^3 \)

\( \frac{dV}{dt} = K \cdot S \), \( K \) is a constant, since the evaporation is proportional to \( S \).

Manipulating \( S \) and \( V \) we see \( S = 4\pi (\frac{3}{4})^{\frac{3}{2}} \sqrt{3} \).

Let's swallow all the numbers with \( K \), since it's just a constant, so \( \frac{dV}{dt} = K \cdot 4\pi (\frac{3}{4})^{\frac{3}{2}} \sqrt{3} \), and since the volume is decreasing, let's put an explicit negative sign in. Thus \( \frac{dV}{dt} = -K \cdot V^{\frac{3}{2}} \).
10.1 #17 A certain drug is administered intravenously into a patient. Fluid containing 5mg/cm$^3$ enters at a rate of 100 cm$^3$/hr. The drug leaves the bloodstream at a rate proportional to the amount present, with constant of proportionality 0.04 hr$^{-1}$.

a) Write a differential equation for the amount of drug in the patient’s bloodstream at time $t$.

$$y' = \text{sources} - \text{sinks} \quad (\text{in} \hspace{1em} \text{mg/hr}),$$

$$y' = 5 \text{mg/cm}^3 \cdot 100 \text{cm}^3/\text{hr} - 0.04 \frac{\text{mg}}{\text{hr}} \cdot y \text{ mg}$$

$$y' = 500 - 0.04y$$

b) After a long time, $y$ approaches the steady state where

$$0 = y' = 500 - 0.04y$$

$$0.04y = 500$$

$$y = \frac{500}{0.04} = 20 \text{ mg}$$

We can see this from the slope-field diagram:
1.2 #13 The charge $Q(t)$ on the capacitor in the circuit satisfies the ODE $R \frac{dQ}{dt} + \frac{Q}{C} = V$.

a) If $Q(0) = 0$, find $Q(t)$ and sketch the solution.

$RQ' + \frac{1}{2}Q = V$

$RQ' = -\frac{1}{2}Q + V = -\frac{1}{2}(Q-CV)$

$\frac{R}{Q-CV} Q' = -\frac{1}{2}$

$\int \frac{R}{Q-CV} dQ = \int -\frac{1}{2} dt$

$R \ln |Q-CV| = -\frac{1}{2} t + k$

$\ln |Q-CV| = -\frac{t}{CR} + k$ (we can absorb $\frac{1}{R}$ into the arbitrary constant $k$)

$Q-CV = ae^{-t/CR}$

$Q(t) = ae^{-t/CR} + CV$

If $Q = Q(0) = a + CV$, then $a = -CV$.

So

$Q(t) = CV(1 - e^{-t/CR})$

If $CV > 0$

I'm not sure what the convention is.

If $CV < 0$
b) \[ \lim_{t \to \infty} Q(t) = CV \]

c) If \( Q(t_1) = CV \), but the battery is removed, we get a new IVP:
\[ \begin{align*}
  RQ' + \frac{Q}{RC} &= 0 \\
  Q(t_1) &= CV
\end{align*} \]

(Since there is no battery, \( V = 0 \))

So \( Q' = -\frac{1}{RC} Q \)

So \( Q(t) = ae^{-t/RC} \)

\( CV = Q(t_1) = ae^{-t_1/RC} \)

So \( a = CVe^{t_1/RC} \)

So \( Q(t) = CVe^{t/RC}e^{-t/RC} = CVe^{(t_1-t)/RC} \)

\( CV \)

\[ \text{---} \]

\[ \to \]
1.3 #29 We want to derive the equation of motion for a pendulum: \( \frac{d^2 \theta}{dt^2} + \frac{g}{L} \sin \theta = 0 \)

\[ a) \]

![Diagram of a pendulum with forces labeled: Tension, mg, normal component of mg, \( \sin \theta \), \( \cos \theta \), and \( \frac{\dot{\theta}}{\theta} \).]

Tension = -(normal component of) \( \frac{mg}{mg} \)

So the sum of the two forces
\[ mg + \text{tension} = \vec{F}. \]

We know
\[ \sin \theta = \frac{\vec{F}}{mg} = \frac{\text{opp}}{\text{hyp}}, \]

so \( \vec{F} = mg \sin \theta \)

\[ b) \vec{F} = ma, \text{ so } \vec{F} = m \text{ tangential} \]

\[ mg \sin \theta = -m \cdot L \frac{d^2 \theta}{dt^2} \]

(since gravity tends to slow the motion of the pendulum upwards)

\[ \text{so } g \sin \theta = -L \frac{d^2 \theta}{dt^2} \]

\[ c) \text{ so } \frac{d^2 \theta}{dt^2} + \frac{g}{L} \sin \theta = 0 \text{ as required.} \]
2.3.4

A 500-gal tank contains 200 gal of H₂O and 100 lb of salt. H₂O w/ 1 lb of salt per gallon enters at a rate of 3 gal/min and the mix flows out at a rate of 2 gal/min. Find the amount of salt.

Find the concentration of salt just before overflow.

Compare with the theoretical limit.

\[ V(0) = 200 \text{ gal} \]
\[ V(t) = 3 \text{ gal/min} \times t - 2 \text{ gal/min} = 1 \text{ gal/min} \]

so \[ V(t) = 1 \text{ gal/min} \times t + 200 \text{ gal} \]

and when \[ V(t) = 500 \text{ gal} = 1 \text{ gal/min} \times t + 200 \text{ gal} \]

\[ 300 \text{ gal} = 1 \text{ gal/min} \times t \]

\[ 300 \text{ min} = t \]

at the time the tank overflows.

Now

\[ a(t) = \frac{C_i \gamma_i - C_o \gamma_o}{C_i \gamma_i - C_o \gamma_o} \left( \frac{dV}{V(t)} \right) \]

\[ = 1 \text{ gal} \cdot 3 \text{ gal/min} - 2 \text{ gal/min} \left[ \frac{3(t)}{1 \text{ gal/min} \times t + 200 \text{ gal}} \right] \]

and \[ a(t) = 3 \text{ gal/min} \times t - 2 \text{ gal/min} \left[ \frac{3(t)}{1 \text{ gal/min} \times t + 200 \text{ gal}} \right] \]

\[ M(t) a(t) = M(t) \left( \frac{2}{1 + 200} \right) a(t) = 3 \mu(t) \]

\[ \frac{dM}{dt} = \mu(t) \]

\[ f(t) = \frac{\mu(t)}{2} \]

so \[ \frac{dM}{dt} \]

\[ M(t) \]

\[ \mu(t) = (t + 200)^2 \]

so

\[ (t + 200)^2 a(t) + (t + 200)^2 \frac{3}{1 + 200} a(t) = 3(t + 200)^2 \]

\[ \frac{d(t + 200)^2 a(t)}{dt} = 3(t + 200)^2 \]

\[ (t + 200)^2 a(t) = (t + 200)^3 + C \]

\[ a(t) = t + 200 + \frac{C}{(t + 200)^3} \]
a(0) = 100 lb of salt so

\[ 100 = 0 + 200 + \frac{c}{(t+200)^2} \]

\[ 100 = 200 + \frac{c}{(200)^2} \]

\[-100(200)^2 = c \]

\[ \Rightarrow a(t) = t + 200 - \frac{100(200)^2}{(t+200)^2} \quad \text{when } t \leq 300 \text{ min} \]

at \( t = 300 \) the tank overflows

\[ a(300) = 300 + 200 - \frac{100(200)^2}{(300+200)^2} \]

\[ = 500 - \frac{100(200)^2}{500^2} \]

\[ = 484 \text{ lbs} \]

there are 500 gal of H2O in the tank at \( t = 300 \text{ min} \)

so

\[ \Rightarrow \text{the concentration is } \frac{484 \text{ lbs}}{500 \text{ gal}} = 0.968 \text{ lbs/gal} \]

If the tank was infinite

as \( t \to \infty \)

the concentration will go to 1\% gal
2.3 #18 Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and those of its surroundings. If a cup of coffee has a temperature of 200°F when poured, and 1 minute later has cooled to 190°F, where the ambient temp is 70°F, when is the coffee 150°F?

First we must set up the equation.

\[ T'(t) = \lambda (T(t) - 70^\circ) \text{ °F/min} \]

so \[ \frac{T'}{T-70} = \lambda. \] Solving, \ln |T-70| = \lambda t + c

so \[ T(t) = ae^{\lambda t} + 70 \]

Now we know \( T(0) = 200 \)

\[ T(1) = 190 \]

so we can solve for \( a \) and \( \lambda \):

\[ 200 = T(0) = ae^{\lambda 0} + 70 = a + 70 \Rightarrow a = 130 \]

\[ 190 = 130e^{\lambda 1} + 70 \Rightarrow \frac{120}{130} = e^\lambda \]

so \( \lambda = \ln \left( \frac{120}{130} \right) \approx -0.08 \)

Thus \[ T(t) = 130e^{-0.08 t} + 70 \]

so \[ 150 = T(t) = 130e^{-0.08 t} + 70 \]

when \[ \frac{80}{130} = e^{-0.08t} \]

\[ t = -\frac{1}{0.08} \ln \left( \frac{80}{130} \right) \approx 6.07 \text{ minutes after the coffee is poured} \]
2.5 b) Semicrystalline Equilibrium Solutions

a) Consider the equation
\[ \frac{dy}{dt} = k(1-y)^2, \quad k > 0. \]
Since \( k(1-y)^2 = 0 \) only when \( y = 1 \), the only equilibrium solution is \( y(t) = 1 \).

b) Since \( y'(t) = \frac{dy}{dt} = k(1-y)^2 > 0 \) for \( y \neq 1 \), all \( t \), \( y(t) \) is an increasing function for all \( t \) if \( y(t) \) is any solution except \( y = 1 \).

\[ \begin{array}{c}
\text{\( f(y) \)} \\
\text{\( y' \)}
\end{array} \]

\[ \begin{array}{c}
\text{\( k \)} \\
\text{\( 0 \)} \\
\text{\( 1 \)} \\
\text{\( 2 \)} \\
\text{\( y \)}
\end{array} \]

c) To solve:
\[ y' = k(1-y)^2 \]
\[ \frac{y'}{(y-1)^2} = \frac{y'}{1-y} = k \]
So \( \int \frac{dy}{(y-1)^2} = \int k \, dt \)

Let \( u = y-1 \) \quad \text{then} \quad \int \frac{du}{u^2} = \int k \, dt \]
\[ du = dy \]
\[ \frac{1}{1-y} = -\frac{1}{u} = k \, dt + C \]

\[ 1 - \frac{1}{u} = \frac{kt+C-1}{kt+C} \]

\[ y = 1 - \frac{1}{kt+C} = \frac{kt+C-1}{kt+C} \]
\[ \frac{y_0}{c} = y(0) = 1 - \frac{1}{c} \] \quad \text{then} \quad \frac{1}{1-y_0} = C \]
\[ \frac{y(t)}{c} = \frac{kt + \frac{1}{1-y_0} - 1}{kt + \frac{1}{1-y_0}} = \frac{(kt-1)(1-y_0) + 1}{kt(1-y_0) + 1} = \frac{(1-y_0)kt + y_0}{(1-y_0)kt + 1} \]
Graph $y(t)$: There is a vertical asymptote

$$a + (1 - y_0)kt + 1 = 0$$

$$1 = (y_0 - 1)kt$$

$$\frac{1}{k(y_0 - 1)} = t$$

So this is $> 0$ if $y_0 > 1$

$< 0$ if $y_0 < 1$

So if $y_0 > 1$, we get:

whereas if $y_0 < 1$, we get:
2.5 #14 Consider the equation \( \frac{dy}{dt} = f(y) \) and suppose \( y \) is a critical point, that is, \( f(y) = 0 \). Show that the constant equilibrium solution \( \phi(t) = y \) is asymptotically stable if \( f'(y) < 0 \) and unstable if \( f'(y) > 0 \).

We know that if \( f(y) > 0 \) for all \( y \in (y_1 - \delta, y_1) \) and \( f(y) > 0 \) for all \( y \in (y_1, y_1 + \delta) \), then \( y_1 \) is an asymptotically stable equilibrium, and that if \( f(y) < 0 \) for all \( y \in (y_1 - \delta, y_1) \) and \( f(y) > 0 \) for all \( y \in (y_1, y_1 + \delta) \) then \( y_1 \) is unstable.

So for the asymptotically stable case, we need to show that if \( f'(y_1) < 0 \), then there is some \( \delta > 0 \) so that if \( y \in (y_1 - \delta, y_1) \) then \( f(y) > 0 \) and if \( y \in (y_1, y_1 + \delta) \) then \( f(y) < 0 \).

Let \( f'(y_1) = -c < 0 \).

By the definition, \( f(y) = \lim_{y \to y_1} \frac{f(y) - f(y_1)}{y - y_1} = \lim_{y \to y_1} \frac{f(y)}{y - y_1} \).

Since \( f(y_1) = 0 \),

So let \( \delta = \frac{c}{2} \). Then by definition of limit, there is some \( \delta \) so that for all \( y \in (y_1 - \delta, y_1 + \delta), y \neq y_1 \),

\[
\frac{f(y)}{y - y_1} \in (-c - \frac{c}{2}, -c + \frac{c}{2}) = (-\frac{3c}{2}, -\frac{c}{2}).
\]

That is, for all \( y \in (y_1 - \delta, y_1 + \delta), y \neq y_1 \),

\[
\frac{f(y)}{y - y_1} = -K < 0
\]

so \( f(y) = -K(y - y_1) \) if \( y < y_1 \), i.e., \( y \in (y_1 - \delta, y_1) \).

This was what we needed. The unstable case is similar.
2.5 #25 Bifurcation Points

Consider an equation of the form
\[ \frac{dx}{dt} = (R - R_c)x - ax^3, \]

where \( R_c \) and \( a \) are fixed positive constants and \( R \) is a parameter which may vary.

(a) If \( R < R_c \), then the equilibrium solutions are \( x \) such that
\[ (R - R_c)x - ax^3 = 0, \]
\[ (R - R_c)x \left(1 - \frac{a}{R - R_c}x^2\right) = 0, \]
where \( \frac{a}{R - R_c} < 0. \)
So \( 1 - \frac{a}{R - R_c}x^2 > 0 \) for \( x \in \mathbb{R} \).
Thus the only equilibrium solution is \( x = 0 \). Since \( f(x) \geq 0 \) for \( x < 0 \), \( f(x) \leq 0 \) for \( x > 0 \), it is stable.

(b) If \( R > R_c \), then \( 1 - \frac{a}{R - R_c}x^2 \) factors as \( (1 - \sqrt{\frac{a}{R - R_c}}x)(1 + \sqrt{\frac{a}{R - R_c}}x) \), so
\[ x = 0, \quad x = \sqrt{\frac{R - R_c}{a}} \quad \text{and} \quad x = -\sqrt{\frac{R - R_c}{a}} \]
are all equilibrium solutions.

(c) Graph of equilibrium \( x \) for various \( R_c \) values:

\[ \text{Note: These are not the same axes as for the slope-field diagram for an autonomous equation (y and y') the graph of a given solution for some fixed x(0) and R (Earyy)} \]