Taylor Series Problems

1. Use the formula for the geometric series to find a Taylor series formula for $1/(1 - x^2)$.
2. Take the derivative of both sides of the geometric series formula. Then plug in $x = 1/2$ to get an interesting formula.
3. Use the fact that $\arctan(1/\sqrt{3}) = \pi/6$ to get a series formula for $\pi$.
4. Derive the formula given for $\ln(1 + x)$ by integrating, as we did to get the series for $\ln(1 - x)$.
5. Let $c$ be a positive number. Consider the formula
   \[
   \frac{1}{x} = \frac{1}{c + x - c} = \frac{1}{c} \cdot \frac{1}{1 + \frac{x-c}{c}}
   \]
   Use geometric series on the last term to find a series formula for $1/x$ about $x = c$.
   (Note: this series can be used to approximate $1/x$ without long division – that’s handy if, for instance, $x$ has a large number of digits.)
6. Write
   \[
   \frac{x}{x^2 - 3} = \frac{x}{3} \cdot \frac{1}{1 - x^2/3}
   \]
   Expand the rightmost factor as a geometric series to get a Taylor series for $x/(x^2 - 3)$.
7. Let $\alpha$ be a real number that is not a positive integer. Show that
   \[
   \lim_{k \to \infty} \left| \frac{\binom{\alpha}{k+1}}{\binom{\alpha}{k}} \right| = 1
   \]
   (This shows that the radius of convergence of the Binomial Series is 1.)
8. Find a Taylor series solution $y$ to the differential equation $y' = 3 \cdot y$ and $y(0) = 2$. Find the radius of convergence of this series.
9. Find a Taylor series solution $y$ to the differential equation $y' = 4 \cdot y - 20$ and $y(0) = 9$.
10. Find a Taylor series solution $y$ to the differential equation $y'' = -y + 1$ and $y(0) = 2$ and $y'(0) = 2$.
11. Find a Taylor series solution $y$ to the differential equation $y'' = -x^2 \cdot y$ and $y(0) = 0$ and $y'(0) = 1$. 
12. Use the Taylor series for $\cos(x)$ to find a series for $\cos(x^3)$. Now find a series for
\[ \int_0^2 \cos(x^3) \, dx \]
(Note: There is no formula for the antiderivative of $\cos(x^3)$ in terms of ordinary functions, and so series is about the only way the integral can be done.)

13. Find a Taylor series for $\arcsin(x)$, using that
\[ \arcsin(x) = \int_0^x \frac{dt}{\sqrt{1 - t^2}} \quad \text{where} \quad -1 \leq x \leq 1 \]
(Hint: $1/\sqrt{1 - t^2} = (1 + (-t^2))^{-1/2}$.)

14. Find a series formula for this important statistical constant:
\[ \frac{1}{\sqrt{2 \cdot \pi}} \cdot \int_{-1}^1 \exp(-x^2/2) \cdot dx \]

15. Find the Taylor series about 3 for the function $x^2 - 2x + 7$. (Hint: as in class, use Taylor’s theorem to find the coefficients.)

16. Use Taylor’s Theorem to determine all the coefficients $a_n$ in the formula
\[ \ln(x) = \sum_{n=0}^{\infty} a_n \cdot (x - 2)^n \]

17. Find the Taylor series for $\sec(x)$ about $x = 0$ up through the $x^3$ term. Do you think you could find a Taylor series for this function?

18. Look at the general Taylor series remainder formula for $\sin(x)$ about $x = 0$.
   (a) Use the fact that $|\cos(t)| \leq 1$ and $|\sin(t)| \leq 1$ to estimate the remainder when $|x| < 3$.
   (b) Find $n$ so that the remainder is less than $10^{-10}$ for $|x| < 3$. 