PROBLEMS FOR MATH 210

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These problems are coordinated with *Introduction to Differential Equations and Linear Algebra*, by Alan Parks, 5th Ed., $\alpha\lambda\alpha\sigma$ Publishing, Appleton, WI. Some terms are introduced in these problems, and so there is an index at the end.

$\Longleftarrow$ Chapter 2 $\Longrightarrow$

Problem 2.1. Find a solution or solution equation to each of the following IVPs. Remember to resolve absolute values. For problem (c), find the domain, as well.

(a) $\dfrac{dy}{dt} = \dfrac{y \cdot (y-7)}{e^{5t}}$ and $y_0 = 5$

(b) $\dfrac{dx}{dt} = \dfrac{\sin(7t) \cdot (x^2 - 16)}{x^2 + 3}$ and $x_0 = 4$

(c) $\dfrac{dz}{dt} = 2 \cdot t \cdot \sec(z)$ and $z_0 = 0$

Problem 2.2. Show that $y = 0$ (constant) and $y = t^3$ are *both* solutions to this IVP:

$$\frac{dy}{dt} = 3 \cdot y^{2/3}$$

Thus, an IVP does not have to have a *unique* solution. We will discuss the profound significance of uniqueness in class. (Note: you are not asked to *find* solutions, but to *verify* solutions – that means plugging in!)

Problem 2.3. Solve the following IVP's. In each case be deliberate about classifying the problem as separable or first order linear. If logarithms occur, write their arguments appropriately without absolute values.

a) $x' + \dfrac{7}{t-5} \cdot x = 2,\ x_0 = 0$

b) $y' + \cot(t) \cdot y = 2 \cdot \cos(t),\ y(\pi/2) = 0$

c) $A' + t^3 \cdot A^2 = 16 \cdot t^3,\ A_0 = 1$

d) $t \cdot u' + 3 \cdot u = t^3,\ u(1) = 1$

e) $y' = \cos(y) \cdot \tan^5(t),\ y_0 = \pi/2$

f) $z' = z^3,\ z_0 = 1$

g) $y' + \dfrac{2t}{t^2+1} \cdot y = t^2,\ y_0 = 1$

h) $q' - 7 \cdot e^t \cdot q = 0,\ q_0 = -1$

i) $w' - \exp(2 \cdot w) \cdot t = 0,\ w_0 = 0$

j) $y' + 5 \cdot y = 20 \cdot t,\ y_0 = 6$

k) $x' + 3 \cdot t^2 \cdot x = t^6,\ x_0 = 1$

λ) $z' = \dfrac{e^z \cdot \sin(3t)}{z},\ z_0 = 1$

Problem 2.4. Solve for x in this IVP, and determine the *domain* – the set of possible values of t. (Hint: x' has to be t times a non-negative number.)

$$\frac{dx}{dt} = t \cdot \sqrt{9 - x^2} \quad \text{and} \quad x_0 = 0$$

Problem 2.5. Solve this problem, using that it first-order linear in x'.

$$x'' - 3 \cdot x' = 2 \cdot e^{3t} \quad \text{and} \quad x_0 = 1,\ x_0' = 1$$

Problem 2.6. Consider the *Bernoulli Equation*: $x' + p(t) \cdot x + q(t) \cdot x^n = 0$. Make the substitution $w = x^{1-n}$ and see what happens.

Problem 2.7. We will show that there is a continuous solution y to this *discontinuous* IVP:

$$y' - 3 \cdot y = \begin{cases} 2 & \text{when } 0 \le t \le 1 \\ 5 & \text{when } 1 < t \end{cases} \quad \text{and} \quad y_0 = 0$$

(a) Solve for y when $0 \le t \le 1$. Take note of $y(1)$.
(b) Solve for y when $t > 1$, using $y(1)$ as initial value.
(c) Note that y is continuous for all $t \ge 0$. Describe the graph of y at $t = 1$.

Problem 2.8. Modify the gravity and air resistance model on p.22: replace $k \cdot v$ by $k \cdot v^2$. (This is an example of a *drag equation* for rockets at high velocity.) Assuming that we keep $v > 0$ (that we are moving down), find the equilibrium. If v_0 starts below equilibrium, what happens as $t \to \infty$?

Problem 2.9. Suppose that a population P has $P_0 > 0$ and grows at a rate proportional to the m-th power of the population. (Exponential growth occurs when $m = 1$.)
a) Suppose that $0 < m < 1$. Show that $P \to \infty$ as $t \to \infty$.
b) Suppose that $1 < m$. Show that $P \to \infty$ as $t \to A$ for some positive number A. (We say that P *blows up in finite time.*) Make sure $A > 0$.

Problem 2.10. Placing a cake in the oven or a soda can in a refrigerator are special cases of the following: an object with no heat source of its own is placed in an environment of constant temperature E. What is called *Newton's law of cooling*[1] asserts that the object's temperature will change at a rate proportional to the difference between E and the object's temperature. Let y stand for the temperature of the object as a function of time t, and assume that $y_0 < E$. Show that $y \to E$ as $t \to \infty$. In the specific case that $E = 350°$F and $y_0 = 70°$F, suppose that $y = 200°$F after half an hour. What is y after an hour and a quarter?

[1] If the temperature of the object is less than E, it will heat up, and *cooling* becomes *heating*. Mathematicians amuse themselves by thinking of heating as *negative cooling*; physically the two processes are obviously very different!

Problem 2.11. In the Lotka-Volterra equation, let $p = q = r = s = 1$, so that the equilibria are $(0, 0)$ and $(1, 1)$. (Not *one rabbit, one wolf* but one unit of population for each!) Graph the solution curve having $x_0 = 1/2$ and $y_0 = 1/2$. (Hint: find C based on the initial conditions. The graph is hard to get: factor $x' = x(1 - y)$ and think about whether x increases or decreases; do the same for y.)

Problem 2.12. In the logistic equation, it is of mathematical interest to consider an initial value $P_0 < 0$. Show that P *blows down* (goes to $-\infty$) in finite time. (Note: make sure you show that the time is positive when $P = -\infty$.)

Problem 2.13. *(A simple economic model.)* Suppose that the price p of a good changes at a rate proportional to the difference between the demand and supply of that good. Assume that demand is a decreasing, linear function of p, and that supply is an increasing linear function of p. Get a first order linear DE out of this. Will a solution approach equilibrium?

Problem 2.14. Modify the loan-payment equation for this problem: we invest m dollars per month in an account that grows at a monthly interest rate r. Solve the equation for P, and determine the *doubling time:* the value of t when $P = 2 \cdot P_0$.

Problem 2.15. Solve the rocket propulsion equation in the (easier!) case that $\rho = 0$. Show that the velocity of the rocket can be made greater than the velocity of the exhaust gasses. (This fact is important in designing a rocket to escape the gravitational pull of a planet.)

Problem 2.16. An object is placed on the x-axis at the point $x_0 > 0$ with zero initial velocity. An inverse square gravitational force acts on the object, pulling it toward the "sun" at the origin. Write $y = x'$ and get a DE by writing x' and y' in terms of x and y. Now use the equation for dy/dx to solve for y as a function of x. Remembering that $y = x'$, separate x and t, write t as an integral in x. (The integral can be done, and if you are ambitious you might try it, but you can leave the integral alone if you wish.)

Problem 2.17. A tank holds 10 gallons of water and the water is capable of holding a great deal of salt. Initially there is 1 gallon of water and no salt in the tank. Then two spigots are turned on, so that water enters the tank at 3 gallons per minute and salt enters at 2 ounces per minute. The tank is constantly stirred, as well, so that the salt mixes thoroughly with the water without changing the volume of the water. When the water and salt begin to enter, a tap opens at the bottom of the tank and lets out 1 gallon of salt-water per minute. Write down a DE that describes the amount of salt in the tank at time t. (Hint: first solve for the amount w of water as a function of time; let the units on salt and water guide you.) Solve the DE you wrote down; how much salt is in the tank when the tank is full?

Problem 2.18. It is a very important physical principle that systems tend to respond to a periodic stimulus in the same frequency as the stimulus. If f is constant, then the function $\cos(2 \cdot \pi \cdot f \cdot t)$ has frequency f. Show that the solution to

$$x' + k \cdot x = \cos(2 \cdot \pi \cdot f \cdot t), \quad x_0 = 0$$

has frequency f, as well. (You can assume that k is a positive constant.)

$\Longleftarrow$ **Chapter 3** $\Longrightarrow$

Problem 3.1. Prove the associative law of matrix addition.

Problem 3.2. Show that scalar multiplication distributes over matrix addition: that

$$\alpha \cdot (A + B) = (\alpha \cdot A) + (\alpha \cdot B)$$

for all $m \times n$ matrices A, B and numbers α.

Problem 3.3. Compute $A \cdot B$ in each of the following. (Notes: in (b), be careful about the size of the result! In (c), the notation R is for the rotation matrix, introduced on p.43.)

a) $A = \begin{pmatrix} 0 & -1 & 2 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 3 & -1 \\ 10 & 1/2 \\ 0 & 2 \end{pmatrix}$ **b)** $A = \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix}$ $B = \begin{pmatrix} -1 & 0 & 2 & 4 \end{pmatrix}$

c) $A = R(1 + \pi/3)$, $B = R(\pi/6 - 1)$ **d)** $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

Problem 3.4. Find all matrices A such that

$$A \cdot \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Problem 3.5. Let b, c be arbitrary numbers with $c \neq 0$ and define

$$A = \begin{bmatrix} b & c \\ (1 - b^2)/c & -b \end{bmatrix}$$

Show that $A^2 = I_2$. (Note: thus, there are infinitely many *square roots of 1* among the matrices.)

Problem 3.6. Show that $I_m \cdot A = A$ for all $m \times n$ matrices A.

Problem 3.7. In each of the following two cases, compute A^2, A^3, and so on until you see a pattern. What is A^k for an arbitrary positive integer k in each case?

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Problem 3.8. Find a 2×1 matrix u *with non-zero complex number entries*, such that $u^T \cdot u = 0$.

Problem 3.9. Find a non-zero matrix A such that

$$A \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Show that there is **no** non-zero matrix B such that

$$B \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 3.10. For each real number c, define

$$f(c) = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

For real numbers c, d, show that $f(c+d) = f(c) + f(d)$ and $f(c \cdot d) = f(c) \cdot f(d)$. Define $i = R(\pi/2)$, and show that $i^2 = f(-1)$. Does this suggest a way to represent the complex numbers using matrices?

Problem 3.11. (Continuing the previous problem.) Define

$$r = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

What real number corresponds to r^2? Use matrices to show that

$$(\sqrt{2} - 1) \cdot (\sqrt{2} + 1) = 1$$

$\Longleftarrow$ Chapter 4 $\Longrightarrow$

Problem 4.1. Use Elimination (by hand) to solve the systems of equations, noting the pivots and elementary operations along the way and remarking the rank and nullity at the end.

a)
$$\begin{aligned} 2A - 2B - 10C &= 4 \\ -2A + 3B + 13C &= -3 \\ 3B + 9C &= 3 \end{aligned}$$

b)
$$\begin{aligned} x_1 - 2x_2 + 2x_3 - 3x_4 &= 19 \\ -3x_1 + 6x_2 - 8x_3 + 13x_4 &= -71 \end{aligned}$$

c)
$$\begin{aligned} p - 2q - r &= -6 \\ 2p - 2q - 2r &= -8 \\ -p + q + 2r &= 3 \\ -p + 2q + 2r &= 5 \end{aligned}$$

d)
$$\begin{aligned} x + z + w &= 2 \\ x - y + 3z - 2w &= -3 \\ 4x - 3y + 10z - 5w &= -5 \end{aligned}$$

Problem 4.2. Find b so that this system is consistent.

$$\begin{aligned} x_1 - 3x_2 + 5x_3 &= 4 \\ 2x_1 + x_2 - 3x_3 &= b \\ 5x_1 + 6x_2 - 14x_3 &= 8 \end{aligned}$$

Problem 4.3. Find all matrices A such that

$$A \cdot \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \cdot A$$

(Hint: what is the size of A? Let its entries be unknowns and solve equations.)

Problem 4.4. Use Elimination to solve the following system of equations (given in terms of its augmented matrix), but perform the arithmetic steps to three significant digits with an exact base-10 exponent. Find the approximate solution this way. Now solve the system exactly. Is there much difference between the approximate and exact solutions?

$$\begin{pmatrix} 5 & 5\cdot 10^5 & 6.18\cdot 10^5 \\ 2\cdot 10^{-5} & 3 & 3.72 \end{pmatrix}$$

Problem 4.5. In each case, find an example of a system of linear equations with the indicated features. (Feel free to use examples from other homework problems or from class.)
a) Same number of equations as variables, infinitely many solutions.
b) Less equations than variables, no solution.
c) Less variables than equations, unique solution.

Problem 4.6. When Elimination is performed on the 2×3 matrix A (as coefficient matrix with no right side), what are the possible row-echelon forms that could result? (Use 1's and 0's where they have to occur; use stars for unknown entries.)

Problem 4.7. (*Leontief's model of an economy*) We have an economy consisting of n people, each of whom produces one unit of a unique product. Person i produces $P[i]$ worth of product. In order to make one unit of product i, person i purchases $A[i,j]$ units of product j. (So that $0 \le A[i,j] \le 1$.) We assume that all of what is produced is purchased. What does this say about the matrix A? What does the matrix $A\cdot P$ measure? Why might it be interesting to know whether there is an $n\times 1$ matrix P such that $A\cdot P = P$? (The existence of P is that the economy is *closed*.)

Problem 4.8. The table below gives the matrix A of the previous problem in a Leontief model. Solve for the price vector P.

	Units purchased each year			
	wheat	bread	wood	tables
wheat producer	0	0.3	0.15	0.2
bread producer	0.8	0.2	0.2	0.3
wood producer	0.1	0.25	0.1	0.4
tables producer	0.1	0.25	0.55	0.1
unit price	P[1]	P[2]	P[3]	P[4]

Problem 4.9. (*Linear Markov process*) Suppose that if it is sunny today, then there is a definite chance p (so that $0 \le p \le 1$) that it will be sunny tomorrow. Suppose that if it is not sunny, it is rainy, so there the chance of rain the day after a sunny day is $1-p$. Suppose that if it is rainy today, the chance it is sunny tomorrow is q, and the chance it is rainy tomorrow is $1-q$. Today it is sunny. What is your forecast for 10 days from now? What happens after a long period of time?

Problem 4.10. (*Polynomial values.*) Given arbitrary coefficients $c_0, c_1, \ldots, c_n$, we can define a polynomial

$$f(x) = c_0 + c_1\cdot x + \cdots + c_n\cdot x^n$$

We say this polynomial has degree *at most* n, since c_n could be 0. We want to notice that $f(x)$ is a matrix product:

$$f(x) = \begin{pmatrix} 1 & x & x^2 & \cdots & x^n \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix}$$

We will see that it will be useful to regard the right column as *variable*. Call it C. Now suppose we have m values $x_1, x_2, \ldots, x_m$. Express the column

$$\begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_m) \end{pmatrix}$$

as $V \cdot C$ where V is $m \times (n+1)$. (The matrix V is called a *Vandermonde matrix*[2] for $x_1, \ldots, x_m$.)

Problem 4.11. (Continuation of the previous problem.) Suppose that $m \geq (n+1)$, and let V be the $m \times (n+1)$ Vandermonde matrix for $x_1, \ldots, x_m$ where the x_j are distinct. Show that $V \cdot X = \mathbb{O}_{m\times 1}$ has a unique solution. (Hint: remember the algebraic fact: a non-zero polynomial of degree at most n can have at most n roots. If $V \cdot C = \mathbb{O}$ with $C \neq \mathbb{O}$, then the polynomial $f(x)$ formed from C is a non-zero polynomial; what are its roots?)

Problem 4.12. Use Elimination to find a polynomial $f(x)$ of degree at most 2 such that $f(2) = 3$ and $f(3) = 10$ and $f'(2) = 7$.

Problem 4.13. (*Kirchoff's Laws.*) The word *graph* is often used for a set of *vertices* (points), some of which are connected by *edges* (curves). A simple electrical circuit is a graph in which the vertices are *junctions* and the edges are connecting wires or components. In each edge there is a *current*[3] J. If we choose a *direction* for each edge by putting an arrow on one end, then the sign of J indicates the direction of the current – with the arrow if $J > 0$ and against it if $J < 0$. *Kirchoff's Current Law*[4] asserts that, at each junction, the sum of the currents coming in (on arrows) is equal to the sum going out. Observe that this is a system of linear equations. A *tree* in a graph is a set of edges that does not contain a loop.[5] In the graph for an electrical circuit, choose a set of edges that forms a tree, using as many edges as possible. In the equation for Kirchoff's Current Law, this set of edges can be a set of

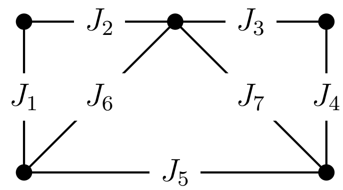
[2] That Vandermonde's name is attached to this matrix is apparently a mistake! In a paper the mathematician Lebesgue claims that the mistake is due to a misreading of the notation used by Vandermonde in one of his papers.

[3] For those unfamiliar with electronics: current measures the number of electrons per unit time passing some point in the wire.

[4] This rule is also called *Kirchoff's Junction Rule*, and by other names as well.

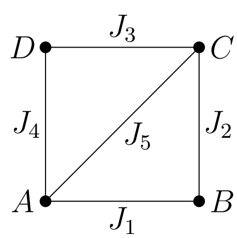
[5] The idea of a loop is fairly intuitive; here is a formal definition: a *loop* is a list $v_1, \ldots, v_k$ of vertices such that there is an edge between v_j and v_{j+1} for each j with $1 \leq j < k$. Also, $v_1 = v_k$ and $k \geq 2$. Thus, a loop travels over edges, ending up where it started. A loop does not have to follow the arrows.

pivoted variables. (The proof involves some elementary graph theory.) Demonstrate this in the case of the circuit here. Note that the currents have been labeled but you will need to choose a direction for each, and you will need to choose a maximal tree. Choose two different trees and, in each case, show that the currents corresponding to edges in your tree can be used as pivots in Elimination. (Hint: in the columns of the augmented matrix, write the tree variables to the left.)



Problem 4.14. (Continuing the previous problem.) In a graph representing an electrical circuit, each edge has an associated *potential drop*[6] This *drop* can be positive or negative. There is a second Kirchoff's Law: the *Voltage Law*[7]. which governs the potential: the sum of the drops around each loop is zero.[8] This is yet another system of linear equations! For each current J_i in the previous graph, define a potential V_i. The tree edges you found in the previous problem give the *free variables* V_i for the voltage law. Write down the equations for the Voltage Law and show that the tree edges can give the free variables. (Hint: write the non-tree edge variables to the left.)

Problem 4.15. Here is another circuit. Get the equations for the Current Law and for the Voltage Law. Think about basic variables in each case.



Problem 4.16. The *Fibonacci numbers* form a sequence $F_0, F_1, \ldots$ defined by recursion: $F_0 = 1$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Show that there is a matrix M such that

$$\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = M \cdot \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} \quad \text{for all} \quad n \geq 0$$

Now show that

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = M^n \cdot \begin{pmatrix} F_0 \\ F_1 \end{pmatrix} \quad \text{for all} \quad n \geq 0$$

[6]Potential is usually measured in volts and represents electromotive force: work done per unit charge. The drop is often called the *voltage drop*.

[7]This rule is also called *Kirchoff's Loop Law*

[8]As we travel around a loop, if potential V is encountered in the direction of an arrow, it counts as $+V$ around the loop; if we are moving against the arrow, then $-V$ is added.

Problem 4.17. We have yellow flags and blue flags that are one foot tall, and we have red flags that are two feet tall. Let G_n be the number of ways to arrange yellow, blue, and red flags on n feet of flag pole. Explain why[9]

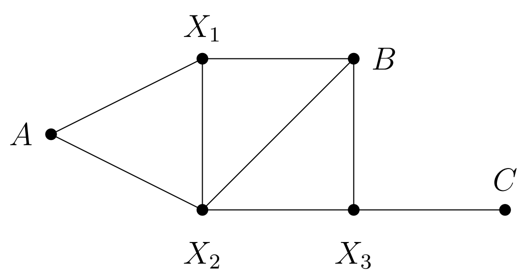
$$G_n = 2 \cdot G_{n-1} + G_{n-2} \quad \text{for} \quad n \geq 3$$

Also explain why $G_1 = 2$ and $G_2 = 5$. Find a matrix N such that

$$\begin{bmatrix} G_{n+1} \\ G_{n+2} \end{bmatrix} = N^n \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{for} \quad n \geq 0$$

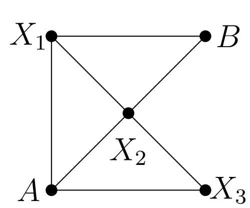
Problem 4.18. (*heat equilibrium*) In the graph below, the nodes are locations where heat is measured. The nodes labeled with single letters are kept at constant heat. The nodes labeled X_j can change as heat diffuses across the edges. We are interested in the equilibrium state where each X_j is constant. Diffusion dictates that each X_j is the average of the temperatures of nodes to which it is connected. Thus, for example,

$$X_1 = \frac{1}{3} \cdot \left[X_2 + A + B \right]$$



Show that the equilibrium equations have a unique solution, given that A, B, C are given constants. (Don't choose values for A, B, C; look at the rank of the coefficient matrix.) Now let $A = 5$ and $B = 2$ and $C = 20$ and solve for the X_j. (Suggestion: choose your pivots to avoid fractions.)

Problem 4.19. Another heat equilibrium problem. Show that if A, B are constants, there is always a unique solution for the X_j.



[9]This problem is a typical problem of *combinatorics* – the counting done in the course of that name.

$\Longleftarrow$ Chapter 5 $\Longrightarrow$

Problem 5.1. For each of the following matrices, determine whether it has an inverse and, if it does, find that inverse.

a) $\begin{pmatrix} -2 & 3 & 5 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ **b)** $\begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$

Problem 5.2. Show that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has rank 2 if and only if $ad - bc \neq 0$. (Hint: Elimination! Consider two cases: first assume that $a \neq 0$, so that number can be used as a pivot. The other case: $a = 0$.)

Problem 5.3. Let A and D be $n \times n$ and invertible. Show that $(AD)^{-1} = D^{-1}A^{-1}$.

Problem 5.4. Let A be invertible and k a positive integer, then $(A^k)^{-1} = (A^{-1})^k$.

Problem 5.5. Let A be invertible. Show that $(A^T)^{-1} = (A^{-1})^T$.

Problem 5.6. Show that if A is 5×8, then there cannot be a matrix C such that $CA = I_8$. (Hint: if C does exist, how many solutions are there to $AX = \mathbb{O}_{5\times 1}$?)

Problem 5.7. Under what circumstances does the following matrix have an inverse?

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

Problem 5.8. Suppose that A is 7×5 and B is 7×1. Could $AX = B$ have a unique solution? Why or why not?

Problem 5.9. Suppose that A is 3×6. Could $AX = B$ have a solution for all possible 3×1 matrices B? Why or why not?

Problem 5.10. Find a 3×4 matrix A and 4×3 matrix C such that $A \cdot C = I_3$. (Note: it is not possible to have $C \cdot A = I_4$.)

Problem 5.11. Consider these data points: $(0, 2), (1, 3), (2, 5), (3, 4)$.

(a) Find the line $y = c_0 + c_1 \cdot x$ of best fit to the points, with its minimal squares error E_1.
(b) Find the parabola $y = c_0 + c_1 \cdot x + c_2 \cdot x^2$ of best fit, with its error E_2. (The numbers c_0, c_1 will be different for the parabola.) Why is it expected that $E_2 < E_1$?

(Note: the calculations are not horrendous by hand, but you might consider using software. Be sure to use Proposition 5.5, in any case.)

Problem 5.12. Suppose we have the equation $g \cdot x = y$, where g is a theoretical constant. Assume we have m data points (x_k, y_k) for $1 \le k \le m$ that have been observed. Find the minimum of the squares error $E(g)$ in this context.[10]

$\Longleftarrow$ Chapter 6 $\Longrightarrow$

Problem 6.1. For each of the following matrices, compute the determinant. Do at least one of them using Elimination, and do at least one using cofactors. (You might want to check your work using a calculator or computer, but do the calculations by hand to make sure you understand the formulas.)

a) $\begin{pmatrix} 2 & 5 & 0 & 0 \\ 1 & 2 & -3 & 1 \\ 3 & 8 & 2 & -1 \\ 1 & 3 & -10 & 3 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & 0 & -4 & 0 \\ 1 & 1 & 3 & -1 \\ 1 & 0 & 6 & -5 \\ 1 & 2 & 1 & 3 \end{pmatrix}$ **c)** $\begin{pmatrix} -2 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 0 & -2 \end{pmatrix}$

Problem 6.2. Consider the general 3×3 matrix A. Compute the determinant using cofactors about two different rows and about one column (your choice), and show that you get the same answer in all three cases. (Of course, this answer should agree with the "crisscross" formula for the determinant of a 3×3 matrix.)

Problem 6.3. Suppose that A and B are $n \times n$ matrices and assume that AB is invertible. Show that A and B are each invertible. (Hint: $\det(A \cdot B)$?)

Problem 6.4. Let A be $n \times n$ and let β be a number. Show that $\det(\beta \cdot A) = \beta^n \cdot \det(A)$. (Hint: $\beta \cdot A$ is obtained from A by a succession of row multiplications.)

Problem 6.5. Complete the following steps to show that if P is the parallelogram with corners at $(0,0)$ and (a,b) and (c,d) and $(a+c, b+d)$, then the area of P is the absolute value of the determinant of

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

a) Let L be the ray from $(0,0)$ through (a,b), and let θ be the angle from this ray to the positive x-axis. Rotate P by the angle θ. Argue that one side of the rotated parallelogram is on the positive x-axis.
b) Recall the rotation matrix $R(\theta)$. Argue that the columns of

$$R(\theta) \cdot A$$

are corners of the rotated parallelogram, and so $R(\theta) \cdot A$ has the form

$$\begin{pmatrix} e & f \\ 0 & g \end{pmatrix}$$

[10] Notes. The minimum is *not the numerical average of y_k/x_k*. Also, the expression $g \cdot x$ is not an arbitrary polynomial of degree at most 1, since there is no constant term, so this is not the kind of regression problem we discussed previously.

c) Show that the area of the rotated parallelogram is $|e \cdot g|$.
d) Complete the following calculation to finish the problem.

$$|e \cdot g| = \left|\det \begin{pmatrix} e & f \\ 0 & g \end{pmatrix}\right| = |\det (R(\theta) \cdot A)| = \cdots$$

$\Longleftarrow$ **Chapter 7** $\Longrightarrow$

Problem 7.1. Solve the following IVP's.

a) $y'' + 2 \cdot y' - 15 \cdot y = 0, \quad y_0 = 5,\ y_0' = 31$
b) $y^{(3)} - 9 \cdot y' = 0, \quad y_0 = 3,\ y_0' = 3,\ y_0'' = 9$
c) $y^{(3)} - y'' - 2 \cdot y' = 0, \quad y_0 = 8,\ y_0' = 2,\ y_0'' = 4$

Problem 7.2. Find the general solution to the following DE's.

a) $y^{(3)} + y'' + 16 \cdot y' + 16 \cdot y = 0$
b) $(D-2)^2 \cdot (D+5)^3 \cdot \left(y^{(3)} + 4 \cdot y'' - 5 \cdot y'\right) = 0$

Problem 7.3. Solve these IVP's.

(a) $y'' + 16 \cdot y = 0, \quad y_0 = -2,\ y_0' = 4$
(b) $y^{(3)} - 3 \cdot y'' + y' - 3 \cdot y = 0, \quad y_0 = 4,\ y_0' = 13,\ y_0'' = 36$
Hint: try 3 as a root.
(c) $y'' - 4 \cdot y' + 7 \cdot y = 0, \quad y_0 = 4,\ y_0' = 8$

Problem 7.4. Solve the following IVP's.

a) $y' - 7 \cdot y = 10 \cdot e^{2t} + 8 \cdot e^{-t}, \quad y_0 = 4$
b) $y'' - 2 \cdot y' - 8 \cdot y = 6 \cdot e^{4t}, \quad y_0 = -2,\ y_0' = -1$
c) $y'' - 5 \cdot y' + 4 \cdot y = 3 \cdot \sin(t) - 5 \cdot \cos(t), \quad y_0 = 0,\ y_0' = 4$
(d) $y'' + y = t^2, \quad y_0 = 4, y_0' = 4$

Problem 7.5. Find the over-general solution to the following DE, and identify the homogeneous part of the solution: $y^{(5)} + 2 \cdot y^{(3)} = \sin(\sqrt{2} \cdot t) + 4t - e^t$.

Problem 7.6. In the underdamped case of the mechanical system, show that $y \to 0$ as $t \to \infty$. Find the frequency with which y oscillates as it goes to 0.

Problem 7.7. The pendulum clock model is Example 3 at the beginning of this chapter. Find the length L in feet given that $g \approx 32$ ft/sec^2, $k \approx 5$, and that the DE is underdamped with period 2 seconds. (This corresponds to a design of Huygens around 1656.)

Problem 7.8. A floating buoy experiences an upward acceleration equal to 11 times the length of its submerged part[11] and a downward constant acceleration of magnitude 9.8 due to gravity. (Acceleration uses length in meters and time in seconds.) Write down the DE governing the submerged length L and find its general solution. (Hint: use the water surface as origin and measure L downward. Up is negative!)

Problem 7.9. (This problem will be used in Chapter 13.) Suppose that $x(t)$ is defined for $0 \leq t \leq 1$ and $x'' + k \cdot x = 0$ there. Assume also that $x(0) = 0$ and $x(1) = 0$ but that $x(t) \neq 0$ for some t between 0 and 1. Show that $k = \pi^2 \cdot n^2$ for some integer n. (Hint: if $k < 0$, show that the solution cannot satisfy the conditions.)

Problem 7.10. (*Reduction of Order*[12]) We consider the equation

$$y'' + p(t) \cdot y' + q(t) \cdot y = g(t) \tag{1}$$

(Since p, q are not constants, the D operator won't help.) Suppose we can find a particular solution $u(t)$ to the related homogeneous equation

$$u'' + p(t) \cdot u' + q(t) \cdot u = 0$$

Set $y = u \cdot v$ where v is an unknown function of t; substitute into (1) and show that you obtain a first order linear equation in v'.

Problem 7.11. Use the technique of the previous problem to solve the DE

$$y'' - \frac{2}{t^2} \cdot y = t^{5/3}$$

given that $u = 1/t$ is a solution to $u'' - (2/t^2) \cdot u = 0$. (Note: don't worry about initial conditions, and feel free to choose particular constants of integration when you need them.)

Problem 7.12. Consider the DE: $t^2 \cdot y'' + 2 \cdot t \cdot y' - 12 \cdot y = 0$. Show that there is a solution of the form $y = t^n$. (Hint: Note that this is **not** a constant coefficient problem, so the operator D is not relevant. Plug in t^n as solution and solve for n.) This equation is called an *Euler-Cauchy DE*[13]

Problem 7.13. Many classes of polynomials that occur in applied problems are defined by DE's. Here are some famous examples; in each case, the subscript n labels a polynomial of degree n. The initial values are different than what we have seen, but they define unique polynomials in each case.

(a) *Chebyshev polynomials*[14] $(1-x^2) \cdot T_n''(x) - x \cdot T_n'(x) + n^2 \cdot T_n(x) = 0$ and $T_n(1) = 1$.
(b) *Hermite polynomials* $H_n''(x) - 2 \cdot x \cdot H_n'(x) + 2 \cdot n \cdot H_n(x) = 0$ and the leading coefficient of $H_n(x)$ is 2^n.

[11]Technical simplification: we're assuming that the mass density of the buoy is 1/11 that of water.

[12]This technique occurs in a paper Euler wrote when he was 21 years old.

[13]I have not been able to find the origin of this equation. Euler and Cauchy lived in different centuries, so it must be the case that each of them considered such an equation independently.

[14]You might find it interesting that $T_n(\cos(\theta)) = \cos(n\theta)$ for each n.

(c) *Legendre polynomials* $(1-x^2)\cdot L_n''(x) - 2\cdot x\cdot L_n'(x) + n\cdot(n+1)\cdot L_n(x) = 0$ and $L_n(1) = 1$.

Find $T_3(x)$ and $H_3(x)$ and $L_3(x)$. (Hint: the D operator is not helpful; in each case you are solving linear equations in the coefficients of the polynomials.)

$\Longleftarrow$ Chapter 8 $\Longrightarrow$

Problem 8.1. Show that the set of differentiable functions on $[0,1]$ is a vector space. (Describe the addition and scalar multiplication and quote a calculus book to show that they obey the properties V1-V7 given for vector spaces. This is basically a reference exercise!)

Problem 8.2. (A crazy example that illustrates the vector space properties.) Let P be the set of positive numbers. For a, b in P, define $a \oplus b = a\cdot b$; for a in P and a real number r, define $r \otimes a = a^r$. Show that P is a vector space using $\oplus$ as addition and $\otimes$ as scalar multiplication. (Note: this is an exercise in very carefully writing the properties (V1)-(V7) for a vector space and translating those properties to real number multiplication and exponentiation.)

Problem 8.3. In each case, show that the given set is a finite dimensional vector space: that it is the set of vectors spanned by a finite set of particular instances.
a) The set of polynomials of degree at most 4 having 3 as a root.
b) The set of 3×2 matrices A such that

$$\begin{pmatrix} 1 & 2 & -1 \\ 4 & 8 & -4 \end{pmatrix} \cdot A = \mathbb{O}_{2\times 2}$$

c) The set of functions $f(t)$ such that $f''(t) + 5 \cdot f'(t) - 14 \cdot f(t) = 0$.

Problem 8.4. In each case, determine whether the first vector given is or is not a linear combination of the remaining vectors in the list.
a) $(-1,1,3)$, $(1,2,3)$, $(4,5,6)$, $(7,8,9)$.
b) x^2+5, x^2-x, $x^2+2\cdot x$, x^2-x+1.
c) $\cos(2x)$, 1, $\sin^2(x)$.

Problem 8.5. Show that *every* 3×1 matrix is a linear combination of the columns of the matrix

$$\begin{pmatrix} 1 & 1 & -3 \\ 0 & -1 & 4 \\ 3 & 2 & -3 \end{pmatrix}$$

Problem 8.6. Let

$$A = \begin{pmatrix} 2 & 4 & 8 \\ 3 & 1 & 2 \end{pmatrix}$$

Which columns of A can be written as a linear combination of the others? (In each case, show how the column is a linear combination of the others.)

Problem 8.7. Let B be a non-zero $m \times 1$ matrix, and let A be an $m \times n$ matrix. Show that the set of solutions to the system $AX = B$ is **not** a vector space.

Problem 8.8. Let V be the (fundamental) vector space of differentiable functions defined on the real numbers, and let W be the set of functions $f(t)$ in V such that $f''(t)$ exists. Show that W is a subspace of V.

Problem 8.9. Let V be the vector space of all polynomials in the variable t. Let W be the set of polynomials $f(t)$ such that

$$\int_0^1 f(t)\cdot dt = 0$$

Show that W is a subspace of V.

Problem 8.10. Let v be a vector in $\mathbb{R}^n$. Define W to be the set of vectors w in $\mathbb{R}^n$ such that $w \circ v = 0$. Show that W is a subspace of $\mathbb{R}^n$.

Problem 8.11. Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^m$. Let w be in $\mathbb{R}^m$ and suppose that $w \circ v_j = 0$ for each j. Show that $w \circ v = 0$ for every v in the span of $v_1, \ldots, v_n$. Show that the set of all such w forms a vector space.

Problem 8.12. Let $v_1, \ldots, v_m$ be the rows of the $m \times n$ matrix A. Let $w \in \mathbb{R}^n$. Show that $w \circ v_j = 0$ for all j if and only if w is in the null space of A.

Problem 8.13. Which of the following are linearly independent?
(a) $(0, 1, 3), (1, 3, 7), (5, 0, 1)$
(b) the polynomials $X^2, X^2 + 3, X^2 - 3$
(c) the functions $\cos(x), \sin(x)$
(d) the columns of an non-invertible 4×4 matrix

Problem 8.14. Let $v_1, \ldots, v_m$ be linearly independent vectors in the vector space V. Let w be a vector in V, and suppose that w is not in the span of $v_1, \ldots, v_m$. Show that $w, v_1, \ldots, v_m$ are independent. (Hint: write a linear combination of all the vectors equal to $\mathbb{O}$, and ask whether the scalar on w is 0, or not.)

Problem 8.15. Find a basis for the vector space spanned by these matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad \begin{pmatrix} -1 & -2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -2 & 9 \\ 6 & 9 & 8 \end{pmatrix}$$

Problem 8.16. Find a basis for the null space of the following matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & -2 & 1 & 3 \\ -5 & 8 & 1 & -1 \end{pmatrix}$$

Problem 8.17. Find a basis for the vector space of solutions to this DE:

$$y''' + 6 \cdot y'' + 8 \cdot y' = 0$$

Problem 8.18. Find a basis for the column space of this matrix.

$$\begin{pmatrix} 1 & 7 & -5 & -1 \\ -4 & 4 & 3 & -1 \\ 23 & 1 & 0 & 2 \end{pmatrix}$$

Problem 8.19. Find a basis for the vector space in $\mathbb{R}^3$ consisting of vectors perpendicular to $(-3, 1, 2)$.

Problem 8.20. Find a basis for the vector space of polynomials of degree at most 4 that have -3 as a root.

Problem 8.21. Let $v_1, \ldots, v_n$ be linearly independent vectors in some vector space. Suppose there are scalars a_j and b_j such that

$$a_1 \cdot v_1 + \cdots + a_n \cdot v_n = b_1 \cdot v_1 + \cdots + b_n \cdot v_n$$

Show that $a_1 = b_1$ and $a_2 = b_2$, and so on. (Hint: bring all terms to the left side.)

Problem 8.22. Suppose that v_1, v_2, v_3 is a basis for the vector space V. Show that

$$v_1 + v_2, \quad v_2 + v_3, \quad v_3$$

is also a basis for V. (Hint: make direct use of both aspects of the definition of *basis*.)

Problem 8.23. Follow the steps given to prove the following fact: let V be a subspace of $\mathbb{R}^n$ and define W to be the set of w in $\mathbb{R}^n$ such that $w \circ v = 0$ for all $v \in V$. We can show that W is a vector space; assume that for now. Let p be the dimension of V, and let q be the dimension of W. Then $p + q = n$.

(a) Let $v_1, \ldots, v_p$ be a basis for V. Write each v_j as $1 \times n$ and form them into the rows of a $p \times n$ matrix A. Then V is the row space of A.
(b) The rank of A is p. (Hint: the rank theorem.)
(c) The set W is the null space of A.
(d) The dimension of W is the nullity of A.
(e) We have $p + q = n$.

$\Longleftarrow$ Chapter 9 $\Longrightarrow$

Problem 9.1. Use the polynomial formula on p.140 to find the eigenvalues of the following matrices. Then find the eigenvectors that belong to those eigenvalues. (As noted, complex eigenvalues occur naturally.)

$$\textbf{a)}\ \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \textbf{d)}\ \begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix}$$

Problem 9.2. Find the characteristic polynomial, eigenvalues and a basis for each of the eigenspaces for these matrices.

$$\textbf{a)}\ \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} -1 & 6 & 7 \\ 0 & 0 & 2 \\ -1 & 6 & 3 \end{pmatrix}$$

Problem 9.3. Find the characteristic polynomial, eigenvalues and and a basis for each of the eigenspaces for these matrices. (Note: The characteristic polynomial for (b) is $\lambda^4 - 3 \cdot \lambda^2 - 2 \cdot \lambda$.)

$$\textbf{a)}\ \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 1 & -1 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 5 & 3 & -3 \\ 1 & 3 & 1 & -1 \\ 1 & 9 & 5 & -5 \end{pmatrix}$$

Problem 9.4. Show that the eigenvalues of the rotation matrix $R(\theta)$ are $\exp(\pm i \cdot \theta)$.

Problem 9.5. Show that the entries on the diagonal of a triangular matrix are the eigenvalues.

Problem 9.6. Let A be $n \times n$, $\lambda \in \mathbb{C}$, let v be $n \times 1$, and suppose that $A \cdot v = \lambda \cdot v$. Show that $A^j \cdot v = \lambda^j \cdot v$ for each positive integer j.

Problem 9.7. Let $f(t)$ be a polynomial, let A be $n \times n$, let λ be an eigenvalue for A, and let v be an eigenvector belonging to λ. Show that $f(A) \cdot v = f(\lambda) \cdot v$.

Problem 9.8. Let A be an $n \times n$ matrix. Then 0 is an eigenvalue of A if and only if A is *not invertible*.

Problem 9.9. Let $n \geq 2$ and let A be $n \times n$ with every entry 1. Show that $0, n$ are eigenvalues of A. (Hint: find eigenvectors explicitly.)

Problem 9.10. Use that $\det(A^T) = \det(A)$ to show that A and A^T have the same characteristic polynomial. (Thus, they have the same eigenvalues.)

Problem 9.11. Suppose that A is $n \times n$ and that each of its columns adds up to the same number λ. Let J be $n \times 1$ having all its entries equal to 1, and compute $A^T \cdot J$ to identify an eigenvalue of A^T. Conclude that A has the same eigenvalue.

Problem 9.12. Let A be an $n \times n$ matrix, each of whose columns adds up to 1. Then there is a non-zero $n \times 1$ matrix P such that $A \cdot P = P$.

Problem 9.13. Recall the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

associated with the Fibonacci sequence. Find an eigenvalue λ for M. Show that $\lambda^{n+2} = \lambda^{n+1} + \lambda^n$. (Hint: use the characteristic polynomial that gives λ as a root.)

Problem 9.14. Let A be $n \times n$ and let $f(t)$ be a polynomial such that $f(A) = \mathbb{O}$. Show that every eigenvalue of A is a root of $f(t)$. (Hint: a previous problem involved $f(A) \cdot v$ where v is an eigenvector for eigenvalue λ.)

$\Longleftarrow$ Chapter 10 $\Longrightarrow$

Problem 10.1. Solve the following IVP, using the eigenvalue/eigenvector method.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Problem 10.2. Solve the following IVP, using the eigenvalue/eigenvector method.

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & -8 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Problem 10.3. Solve the following IVP, using the eigenvalue/eigenvector method.

$$\frac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ -3 & -7 & -2 \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \quad \text{and} \quad X(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Problem 10.4. Show that

$$\exp\left(R(\pi/2) \cdot t\right) = R(t)$$

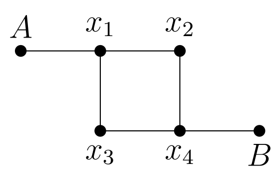
(Hint: the powers of $R(\pi/2)$ repeat in a cycle of 4.)

Problem 10.5. In the albumin model discussed above, replace the constant rate r by an oscillating rate: $74 \cdot \sin(2t)$, where t is time in days. Assume that $c_1 = 1$, $c_2 = 2$, $c_3 = 3$. Find a specific solution of the form $x = A_1 \cdot \cos(2t) + B_1 \cdot \sin(2t)$ and $y = A_2 \cdot \cos(2t) + B_2 \cdot \sin(2t)$, where A_1, B_1, A_2, B_2 are constants. (The general solution will decay toward this specific solution.)

Problem 10.6. In the model of the two masses held by three springs, let $m_1 = 1$, $m_2 = 2$, $k_0 = 1$, $k_1 = 4$, $k_2 = 2$, $h_0 = 1$, $h_1 = 2$, $h_2 = 1$. Find the equilibrium. (The answer will involve c.) Show that the force of spring 0 on mass 1 is the same as the force of spring 1 on mass 2.

Problem 10.7. We have three chemicals A, B, C in a common medium. Substance A turns into substance B at a rate proportional to the amount of A; substance B turns into C at a rate proportional to the amount of B; substance C turns into A at a rate proportional to the amount of C. Find a system of DE's describing the amounts of each chemical. Show that 0 is an eigenvalue of the coefficient matrix, and show that its eigenvectors are equilibria.

Problem 10.8. The picture below depicts an energy grid, as in the problem on p.9. The numbers A, B are constant temperatures, and the x_j change in time. The rate of x_j is equal to the sum of the differences $y - x_j$ where y is a node connected to x_j. (So y can be A or B or one of the other x_k.) Write down the system of DE's that arises. You might be interested in the eigenvalues of the coefficient matrix; we suggest using numerical software to estimate them. You will see that they are negative, and it follows that the solutions drift toward equilibrium.



$\Longleftarrow$ Chapter 11 $\Longrightarrow$

Problem 11.1. Let P_2 be the vector space of polynomials of degree at most 2. Compute $G^{-1}(2 \cdot t + 3 \cdot t^2)$ where G is the coordinate transformation for each of the bases listed. (Thus, you will compute a different $G^{-1}(2 \cdot t + 3 \cdot t^2)$ each time.)
a) Use the basis $1, t, t^2$;
b) Use the basis $1 + t, 1 - t, t + t^2$;
c) Use the basis $1 - t, t + t^2, 1 + t$.

Problem 11.2. Let $f : U \to V$ and $g : U \to V$ be linear transformations (so we are asserting that U, V are vector spaces). Define the function $f+g$ in the usual way: $(f + g)(u) = f \cdot u + g \cdot u$ for all u in U. Show that $f + g$ is a linear transformation.

Problem 11.3. Let $f : U \to V$ be a linear transformation and let α be a number. Define $\alpha \cdot f$ as a function from U to V by $(\alpha \cdot f) \cdot u = \alpha \cdot (f \cdot u)$ for all u in U. Show that $\alpha \cdot f$ is a linear transformation. (Note: this problem and the previous one go a long way toward the following fact: the set of linear transformations from vector space U to vector space V forms a vector space!)

Problem 11.4. Let P_3 be the set of polynomials of degree at most 3, and P_2 those of degree at most 2. Let $D : P_3 \to P_2$ be differentiation, and let $J : P_2 \to P_3$ be defined by $J \cdot f(t) = \int_1 f(t) \cdot dt$. Show that $D \cdot J \cdot f(t) = f(t)$ for all $f(t)$ on P_2. Find $g(t)$ in P_2 such that $J \cdot D \cdot g(t) \neq g(t)$.

Problem 11.5. (Continuing the previous problem.) Use the natural bases for P_3 and P_2, and find the matrix $\hat{D}$ that represents D and the matrix $\hat{J}$ that represents J. Show that $\hat{D} \cdot \hat{J} = I_3$ and that $\hat{J} \cdot \hat{D} \neq I_4$.

Problem 11.6. Let V be the vector space of functions $(a + b \cdot t + c \cdot t^2) \cdot e^{3t}$, where a, b, c are arbitrary real numbers. Find a matrix that represents taking the derivative on V.

Problem 11.7. Consider the operator polynomial $Q = D^2 + D - 6$. Let V be the vector space of functions of the form $(a + b \cdot t) \cdot e^t$, where a, b are arbitrary real numbers. Show that if v is in V, then $Q \cdot v$ is in V. Find a representing matrix for Q and show that it has an inverse. Use the matrix inverse to solve the equation

$$y'' + y' - 6 \cdot y = t \cdot e^t \quad \text{which is} \quad Q \cdot y = t \cdot e^t$$

Problem 11.8. Let L_0, L_1, L_2 be the Legendre polynomials found on page 13. Use these polynomials as a basis for the space P_2 of polynomials of degree at most 2. Let $D : P_2 \to P_2$ be the differential operator, as usual. Find a matrix A that represents D in this context. Show that $A^3 = \mathbb{O}_{3\times 3}$. (This last equation says that the third derivative of a quadratic polynomial is 0.)

Problem 11.9. Let $r : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection about the line $y = -2 \cdot x$. Find the 2×2 matrix A such that $A \cdot p = r \cdot p$ for all p in $\mathbb{R}^2$. Show that $A^2 = I_2$.

Problem 11.10. Let V_3 be the vector space of functions $(a + b \cdot t + c \cdot t^2) \cdot e^{-t}$, where a, b, c are arbitrary real numbers. Let V_2 be the vector space of functions $(a + b \cdot t) \cdot e^{-t}$. Show that the operator polynomial $D + 1$ maps V_3 to V_2. Using the natural bases of V_3 and V_2, find the matrix that represents $D + 1$.

Problem 11.11. Let $L : U \to V$ be a linear transformation. Define W to be the set of u in U such that $L \cdot u = \mathbb{O}_V$. The set W is called the *kernel* of a linear transformation of L. Show that W is a subspace of V. (How do you tell that a subset is a subspace? A proposition in the text on p.121 is relevant.)

Problem 11.12. Let $L : V \to V$ be a linear transformation (note that the vector space V is mapped to itself). For a real number λ, we say that λ is an *eigenvalue* of L if there is a non-zero vector v such that $L \cdot v = \lambda \cdot v$. Let A be a matrix that represents L. Show that A's real number eigenvalues are eigenvalues of L.

Problem 11.13. Let V be the vector space spanned by e^{2t}, $t \cdot e^{2t}$, and let $D : V \to V$ be differentiation, considered as a linear transformation. Find the eigenvalues and eigenvectors for D.

$\Longleftarrow$ Chapter 12 $\Longrightarrow$

Problem 12.1. Go back to all the matrices for which we have computed eigenvalues and eigenvectors; determine which matrices are d'ble and which are not. (Note: use the second test, which involves looking up the dimension of each eigenspace and comparing to the algebraic multiplicity.)

Problem 12.2. Let D be a diagonal matrix. Show, for each positive integer k, that D^k is diagonal, with $D^k[j,j] = (D[j,j])^k$ for each j.

Problem 12.3. Let P be an invertible $n \times n$ matrix and let E be an $n \times n$ matrix. Show, for each positive integer k, that

$$\left(P^{-1} \cdot E \cdot P\right)^k = P^{-1} \cdot E^k \cdot P$$

Problem 12.4. (Continuation of the previous problem.) Show that

$$P^{-1} \cdot \exp(E) \cdot P = \exp(P^{-1} \cdot E \cdot P)$$

Problem 12.5. (Continuation of the previous.) Let

$$E = \begin{pmatrix} -5 & 2 \\ 10 & 3 \end{pmatrix}$$

Use eigenvectors to find a 2×2 matrix P such that $P^{-1} \cdot E \cdot P$ is diagonal. Now use the previous problem to compute $\exp(E)$.

Problem 12.6. Let A be a d'ble $n \times n$ matrix such that $|\lambda| < 1$ for every eigenvalue λ of A. Show that $A^k \to \mathbb{O}_{n\times n}$ as $k \to \infty$. (Hint: Use the previous problem. Note: the limit holds even if A is not d'ble, but it's harder to prove.)

Problem 12.7. Recall that an $n \times n$ matrix A with real entries is *symmetric* if $A^T = A$. It turns out that every symmetric matrix is d'ble. Show that this is true for an arbitrary 2×2 matrix. (Hint: if the eigenvalues are distinct, the matrix is d'ble. How could there be a repeated eigenvalue?)

Problem 12.8. Let

$$F = \frac{1}{3} \cdot \begin{pmatrix} 1 & -4 & 4 \\ -2 & -1 & 4 \\ -2 & -4 & 7 \end{pmatrix}$$

Compute the limit of F^n as $n \to \infty$. (Hint: if λ is an eigenvalue of M, then $\lambda/3$ is an eigenvalue of $M/3$, and $M/3$ has the same eigenvectors as $M/3$.)

$\Longleftarrow$ **Chapter 13** $\Longrightarrow$

Problem 13.1. (One-dimensional diffusion with insulated ends.) Given L, c and $f(x)$, solve

$$\begin{aligned} U_t &= c^2 \cdot U_{xx} \\ U(x,0) &= f(x) \quad \text{for} \quad 0 \le x \le L \\ U_x(0,t) &= 0 = U_x(L,t) \quad \text{for} \quad t \ge 0 \end{aligned}$$

Problem 13.2. (The plucked string problem associated with the one-dimensional wave equation.) Given positive constants L, c and a differentiable function $f(x)$ on $[0, L]$, use separation of variables to find $U(x, t)$, defined for $0 \leq x \leq L$ and $t \geq 0$, such that

$$\begin{aligned} U_{tt} &= c^2 \cdot U_{xx} \\ U(x,0) &= f(x) \quad \text{for} \quad 0 < x < L \\ U_t(x,0) &= 0 \quad \text{for} \quad 0 < x < L \\ U(0,t) &= U(L,t) = 0 \quad \text{for} \quad t \geq 0 \end{aligned}$$

Problem 13.3. (A two-dimensional diffusion problem.) Let L, c be positive constants. Let $f(x, y)$ be defined on the square (x, y) with $0 \leq x \leq L$ and $0 \leq y \leq L$. Find $U(x, y, t)$, where $0 \leq x \leq L$ and $0 \leq y \leq L$, and $t \geq 0$ such that

$$\begin{aligned} U_t &= c^2 \cdot \Big[U_{xx} + U_{yy}\Big] \\ U(x,y,0) &= f(x,y) \\ U(0,y,t) &= U(L,y,t) = U(x,0,t) = U(x,M,t) = 0 \end{aligned}$$

Problem 13.4. (Laplace's equation on a disk.[15]) We are given a differentiable and 2π-periodic function $f(\theta)$. Find $W(r, \theta)$, defined for $0 \leq r \leq 1$ and all θ such that

$$\begin{aligned} W(r, \theta + 2\pi) &= W(r, \theta) \\ r^2 \cdot W_{rr} + r \cdot W_r + W_{\theta\theta} &= 0 \\ W(1, \theta) &= f(\theta) \end{aligned}$$

(Recall that the PDE for W is the polar version of Laplace's equation.)

Problem 13.5. Let n be an integer. Show that $U_n(x, y) = (x + iy)^n + (x - iy)^n$ is harmonic[16] (that it satisfies Laplace's PDE). (Note: $i^2 = -1$, as usual. Assume you can use the usual differentiation rules even with i. You might like to compute U by multiplying out in the cases $n = 1, 2, -1$.)

Problem 13.6. (This problem shows that we cannot necessarily take the derivative of a Fourier series term by term, the way we do for Taylor series.) Find a sine series for the function x on $[0, 3]$. Take the derivative of the sine series term by term, and plug in $x = 3/2$; show that the resulting series cannot converge. (Hint: the Divergence Test.)

Problem 13.7. Find a cosine series for the function $1 - x^2$ on $[0, 2]$.

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[15] This is another *Dirichlet Problem*.

[16] If a_n is a sequence with radius of convergence r, then $U(x, y) = \sum_{n=0}^{\infty} a_n \cdot U_n(x, y)$ defines a harmonic function inside the circle $x^2 + y^2 = r^2$. Thus, every such sequence defines a solution to Laplace's equation. That's a lot of solutions!

Index