The Vector Norm

**Definition** Let $V$ be a vector space. A *norm* on $V$ is a real-valued function $|| \cdot ||: V \to \mathbb{R}$ that satisfies the following properties.

1. $||v|| \geq 0$ for all $v \in V$ and $||v|| = 0$ if and only if $v = 0$.
2. $||\alpha v|| = |\alpha| \ ||v||$ for all scalars $\alpha$ and all vectors $v \in V$.
3. $||u + v|| \leq ||u|| + ||v||$ for all $u, v \in V$.

A vector $v$ is said to be a *normal vector* if $||v|| = 1$.

The Inner Product

**Definition** Let $V$ be a real vector space. A (real) inner product on $V$ is a function $(\cdot, \cdot)$ that maps pairs of vectors from $V$ to real numbers that satisfies the following properties.

1. $(u,v) = (v,u)$ for all vectors $u$ and $v$ in $V$.
2. $(\alpha u + \beta v, w) = \alpha (u, w) + \beta (v, w)$ and $(w, \alpha u + \beta v) = \alpha (w, u) + \beta (w, v)$ for all vectors $u, v,$ and $w$ in $V$ and all real numbers $\alpha$ and $\beta$.
3. $(u,u) \geq 0$ for all vectors $u \in V$ and $(u,u) = 0$ if and only if $u = 0$.

**Definition** Two vectors $u$ and $v$ in a vector space are said to be *orthogonal* with respect to an inner product if $(u,v) = 0$.

**Examples**

The standard inner product on $\mathbb{R}^n$ is the vector dot product.

$$(u,v) = \sum_{i=1}^{n} u_i \ v_i$$

The standard norm on $\mathbb{R}^n$ is

$$||u|| = \sqrt{(u,u)}$$

The vector space $C[0,1]$ of continuous functions on the interval $[0,1]$ has an inner product

$$(f,g) = \int_{0}^{1} f(x) \ g(x) \ dx$$

This inner product is known as the $L^2$ inner product. Likewise, we can define an $L^2$ norm for this vector space by

$$||f|| = \sqrt{\int_{0}^{1} (f(x))^2 \ dx}$$
Other possible norms include the $L^1$ norm

$$\| f \| = \int_0^1 |f(x)| \, dx$$

and the $L^\infty$ norm

$$\| f \| = \max_{x \in [0,1]} |f(x)|$$

Orthogonal Bases

**Definition** A basis $v_1, v_2, \ldots, v_n$ for a vector space $V$ is an orthonormal basis if $(v_i, v_j) = 0$ for all $i \neq j$ and $(v_i, v_i) = 1$ for all $i$.

**Observation** If a vector space has an orthonormal basis, computing coordinate representations with respect to that basis is very easy. Given an arbitrary vector $v$ in $V$, we seek to compute a coordinate vector $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = v$$

If the basis is orthonormal, we can easily compute the coordinates $c_i$ by taking the inner product with respect to $v_i$ on both sides of the equation:

$$(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n, v_i) = (v_i, v_i)$$

$$c_1 (v_1, v_i) + c_2 (v_2, v_i) + \cdots + c_n (v_n, v_i) = (v, v_i)$$

$$c_1 0 + c_2 0 + \cdots + c_i 1 + \cdots + c_n 0 = (v, v_i)$$

$$c_i = (v, v_i)$$

Constructing an Orthonormal Basis

The Gram-Schmidt algorithm is an algorithm that can convert a basis for a vector space into an alternative basis that is orthonormal. Here is an outline of that algorithm. Let $v_1, v_2, \ldots, v_n$ be a basis for a vector space $V$.

1. Convert the vector $v_1$ into a normal vector by dividing it by its own norm.

$$u_1 = \frac{1}{\|v_1\|} v_1$$

2. Construct

$$p_2 = v_2 - (u_1, v_2) u_1$$
The term \((u_1, v_2)u_1\) is the projection of \(v_2\) onto \(u_1\). By construction, \(p_2\) is orthogonal to \(v_1\) (why?).

3. We then form

\[
u_2 = \frac{1}{||p_2||} p_2
\]

in order to make \(u_2\) be both normal and orthogonal to \(u_1\).

4. Next, compute

\[
p_3 = v_3 - (u_1, v_3)u_1 - (u_2, v_3)u_2
\]

and subsequently

\[
u_3 = \frac{1}{||p_3||} p_3
\]

to produce a vector that is normal and perpendicular to both \(u_1\) and \(u_2\).

5. The process repeats until all of the original \(v_i\) vectors have been processed. The result is a set of \(u_i\) vectors which form an orthonormal basis for \(V\).

**The Projection Theorem**

Here is a theorem from the text which also makes use of the concept of a projection.

**Projection Theorem** Let \(V\) be a vector space with an inner product. Let \(W\) be a finite dimensional subspace of \(V\) and let \(v\) by an arbitrary vector in \(V\).

1. There is a unique \(u\) in \(W\) such that

\[
||v - u|| = \min_{w \in W} ||v - w||
\]

\(u\) is known as the projection of \(v\) onto the subspace \(W\).

2. \((v-u,z) = 0\) for all \(z \in W\).

3. If \(\{w_1, w_2, ..., w_n\}\) is a basis for \(W\) then

\[
u = \sum_{i=1}^{n} x_i w_i
\]

where

\[
G x = b
\]

\[
G_{ij} = (w_i, w_j)
\]

\[
b_i = (w_i, v)
\]

The matrix \(G\) is known as the Gram matrix and the equations \(G x = b\) are known as the normal
equations.

4. If \( \{w_1, w_2, ..., w_n\} \) is an orthogonal basis for \( W \) then

\[
u = \sum_{i=1}^{n} \left( \frac{w_i \cdot v}{(w_i, w_i)} \right) w_i\]

**Observation** A very important thing to note about the projection theorem is that the original vector space \( V \) does not have to be a finite dimensional space. The only requirement in the theorem is that \( W \) must be a finite dimensional subspace of \( V \).

This opens an intriguing possibility. Suppose we have a linear operator \( f \) that maps \( V \) to \( V \). If we want to make a finite representation for \( f \) we might do the following:

1. For a \( v \in V \) we compute the projection \( u \) of \( v \) onto \( W \).

2. We compute \( f(u) \) and hope that \( f(u) \) stays in \( W \). If it does not, we project \( f(u) \) back onto the subspace \( W \) to make a vector \( y \).

3. What we have constructed is a restriction of the operator \( f \) onto the subspace \( W \). If \( f \) is still linear on \( W \), we can construct a finite representation for the restricted operator and eventually represent that as a matrix \( A \) such that

\[
A \ u = y
\]