Two Fundamental Theorems about the Definite Integral

These lecture notes develop the theorem Stewart calls The Fundamental Theorem of Calculus in section 5.3. The approach I use is slightly different than that used by Stewart, but is based on the same fundamental ideas.

1 The definite integral

Recall that the expression

$$\int_a^b f(x)\,dx$$

is called the definite integral of $f(x)$ over the interval $[a,b]$ and stands for the area underneath the curve $y = f(x)$ over the interval $[a,b]$ (with the understanding that areas above the x-axis are considered positive and the areas beneath the axis are considered negative).

In today's lecture I am going to prove an important connection between the definite integral and the derivative and use that connection to compute the definite integral. The result that I am eventually going to prove sits at the end of a chain of earlier definitions and intermediate results.

2 Some important facts about continuous functions

The first intermediate result we are going to have to prove along the way depends on some definitions and theorems concerning continuous functions. Here are those definitions and theorems.

The definition of continuity

A function $f(x)$ is continuous at a point $x = a$ if the following hold

1. $f(a)$ exists

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

A function $f(x)$ is continuous in an interval $[a,b]$ if it is continuous at every point in that interval.

The extreme value theorem

Let $f(x)$ be a continuous function in an interval $[a,b]$. The exist numbers m and M such that

$$m \leq f(x) \leq M$$

for all x in $[a,b]$. Furthermore, there are numbers c and d in $[a,b]$ such that $f(c) = m$ and $f(d) = M$.

The intermediate value theorem

Let $f(x)$ be a continuous function in an interval $[a,b]$, and let N be any number between $f(a)$ and $f(b)$. Then there is a number c in $[a,b]$ such that $f(c) = N$.

3 A preliminary result about the definite integral

Theorem Let $f(x)$ be a continuous function on the interval $[a,b]$. Then there exists a c in $[a,b]$ for which

$$f(c)\,(b - a) = \int_a^b f(x)\,dx$$

This theorem essentially says that if you take the area under $f(x)$ over the interval $[a,b]$ and 'flatten it out', you get a rectangle whose height is given by the value of the function at some point c in the interval.

Proof Because $f(x)$ is continuous on $[a,b]$, the extreme value theorem tells us it must have a maximum and a minimum somewhere in that interval.

$$m \leq f(x) \leq M$$

This leads to the inequality for areas

$$m\,(b - a) = \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M\,(b - a)$$

or

$$m \leq \frac{\int_a^b f(x)\,dx}{b - a} \leq M$$

Since $f(x)$ is continuous, by the Intermediate Value Theorem it takes every possible value between m and M. In particular, there is at least one place c at which the function $f(x)$ has a value equal to

$$f(c) = \frac{\int_a^b f(x)\,dx}{b - a}$$

Multiplying both sides by $b - a$ proves the result.

4 The first fundamental theorem of integral calculus

We are now in a position to prove our first major result about the definite integral. The result concerns the so-called area function

$$F(x) = \int_a^x f(t)\,dt$$

and its derivative with respect to x. The only way to compute this derivative is via the definition:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

Since the numerator in the fraction is the difference of two areas, we can rewrite the limit

$$= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} \tag{1}$$

Recall the theorem we just proved.

$$f(c) = \frac{\int_a^b f(x)\,dx}{b - a}$$

If we apply that theorem to the fraction in (1) with

$$a = x$$

$$b = x+h$$

3

$$b - a = h$$

we get

$$\lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h \to 0} f(c)$$

where c is somewhere in the interval $[x, x+h]$. In the limit as h goes to 0, c gets squeezed down to x. Because $f(x)$ is continuous we have that

$$\lim_{h \to 0} f(c) = \lim_{c \to x} f(c) = f(x)$$

The bottom line is that

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

5 The second fundamental theorem of integral calculus

We are now in a position to prove the final and most important theorem in this sequence of results.

Theorem Let $f(x)$ be a continuous function on the interval $[a,b]$. Let $G(x)$ be any function with the property that

$$G'(x) = f(x)$$

Then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

Proof As above, we introduce the area function

$$F(x) = \int_a^x f(t)\,dt \tag{2}$$

By the first fundamental theorem,

$$F'(x) = f(x)$$

Since $F(x)$ and $G(x)$ have the same derivative, they must differ by a constant (Corollary 4.2.7 on page 294).

4

$$F(x) = G(x) + C \qquad (3)$$

What is that constant? Evaluating equations (2) and (3) at $x = a$ gives us

$$F(a) = G(a) + C = \int_a^a f(t)\,dt = 0$$

which leads to

$$G(a) + C = 0$$

$$C = \text{-}G(a)$$

Similarly, evaluating the equations at $x = b$ gives

$$F(b) = G(b) + C = \int_a^b f(t)\,dt$$

or

$$\int_a^b f(t)\,dt = G(b) + C = G(b) \text{ - } G(a)$$

and we are done.