

Work

In physics, work is defined as motion through a distance against a force.

$$\text{work} = \text{force} * \text{distance}$$

If the force is constant throughout the motion, the calculation is very simple. In many applications, however, the force is not constant. In that case, we can take advantage of the fact that over very small distances the force is approximately constant. To compute the total work in moving over a certain distance, we can use an integral to add up the bits of work.

$$W = \int_a^b f(x) dx$$

Examples from the text

In class I showed examples 3 and 4 from the text. You should review those examples carefully.

How much work does it take to build a pyramid?

The great pyramid at Giza is a pyramid with a square base of side length 755 feet and a height of 481 feet. The pyramid is constructed from roughly 2.3 million stones weighing an average of 5000 pounds each. How much energy did it take to pile up the stones in the pyramid?

The best way to approach this problem is to solve it in the abstract first, and then mix in the specific measurements. Consider a pyramid with a square base of length b and a height h . The pyramid has a density of ρ pounds per cubic foot. Suppose we make a thin slice through the pyramid at height y above the base. Since the base of the pyramid is square, the slice will be a square of side length x . Since all of the material in that slice is at the same height, the amount of work needed to lift that slice into position is

$$\rho (\text{volume of slice}) y = \rho (x^2 dy) y \tag{1}$$

A simple argument with similar triangles shows that

$$x = \frac{b}{h} (h - y)$$

Thus, the total work needed to build the pyramid is

$$\int_0^h \rho \frac{b^2}{h^2} (h - y)^2 y dy = \rho \frac{b^2}{h^2} \int_0^h h^2 y - 2 h y^2 + y^3 dy$$

$$= \rho \frac{b^2}{h^2} \left(\frac{h^2}{2} y^2 - \frac{2 h}{3} y^3 + \frac{1}{4} y^4 \right) \Big|_0^h = \frac{1}{12} \rho b^2 h^2 \quad (2)$$

The ρ factor here stands for weight per unit volume. The calculation of the volume of the pyramid is almost identical to the work calculation here. The only difference is that the volume calculation has no density term and is missing the extra factor of y in (1).

$$\text{volume} = \int_0^h \frac{b^2}{h^2} (h - y)^2 dy = \frac{1}{3} \frac{b^2}{h^2} (y - h)^3 \Big|_0^h = \frac{1}{3} b^2 h$$

We are now ready to bring in the specific numbers for this pyramid. The density is the total weight divided by the volume:

$$\rho = \frac{5000 (2.3 \cdot 10^6)}{\frac{1}{3} b^2 h}$$

Plugging this into (2) and setting $h = 481$ gives

$$\text{work} = \frac{1}{12} \left(\frac{5000 (2.3 \cdot 10^6)}{\frac{1}{3} b^2 h} \right) b^2 h^2 = \frac{5000 (2.3 \cdot 10^6)}{4} 481 = 1.38 \cdot 10^{12} \text{ ft-pounds}$$

The unit of work in the metric system is called a Joule. One foot-pound equals 1.356 Joules, so in Joules the energy needed is

$$1.356 (1.38 \cdot 10^{12}) = 1.87128 \cdot 10^{12} \text{ Joules}$$

By way of comparison, the energy content in a liter of jet fuel is approximately $35.5 \cdot 10^6$ Joules. A Boeing 747 has a fuel capacity of 183,380 liters. Thus, the energy content in a fully loaded 747 is $6.51 \cdot 10^{12}$ Joules, or enough to pile up three great pyramids.