

The Integral Test

Introduction to integral comparisons

The method for computing the value of a series is based on taking the limit of a sequence of partial sums.

$$s_n = \sum_{k=0}^n a_k$$

$$\sum_{k=0}^{\infty} a_k = \lim_{n \rightarrow \infty} S_n$$

The major drawback to this method is that it is usually very difficult to derive a formula for the partial sum. To work around this problem, we are going to see a number of more indirect methods for determining whether or not series converge and estimating their values in those cases in which they do converge.

The first such method is the method of integral comparisons.

Recall that we first introduced the integral as a means to solve the problem of finding the area under a curve. Recall also that our original approach for solving the area problem was to approximate the area with a set of rectangles and add up the areas of the rectangles.

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N f(c_n) \Delta x_n = \text{area} = \int_a^b f(x) dx$$

This suggests that there is a connection between integrals and series. Today we are going to see a comparison method that seeks to exploit this connection.

Here is how the method works:

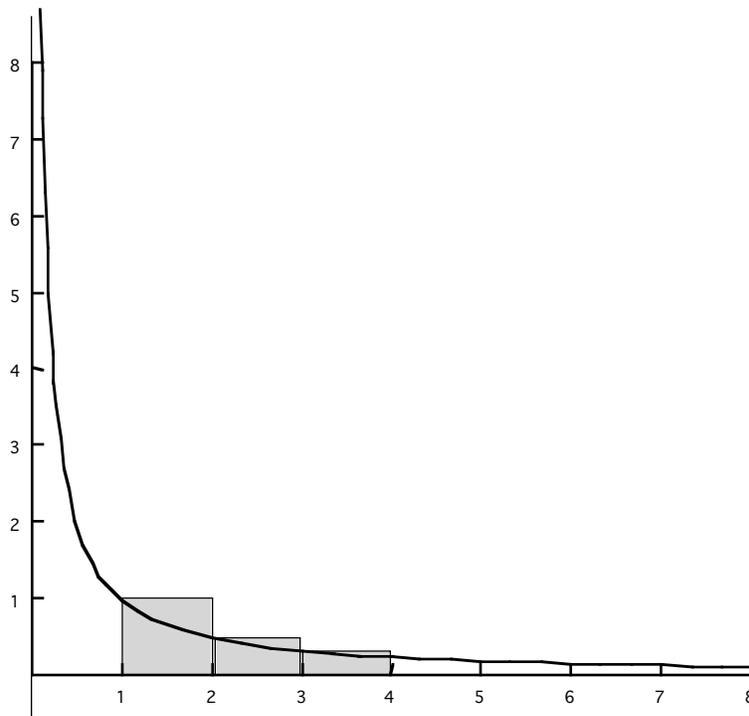
1. Given a series, interpret that series as the sum of a collection of rectangles with width 1.
2. Write down an integral that describes an area that is either definitely smaller or larger than the area covered by the rectangles.
3. Evaluate the integral and say what the result of the integral tells you about the sum.

The harmonic series

We have already shown by another means that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges to $+\infty$. We are going to use the method of integral comparisons to show once again that this series diverges. The picture below shows a set of rectangles superimposed on the graph of $y = 1/x$.



The rectangles are set up so that the height of each rectangle is given by the value of the function $1/x$ at the left endpoint of the interval the rectangle sits on. Clearly the rectangles overlap the curve, so we have

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + 1/2 + 1/3 + \dots = \text{areas of rectangles} > \int_1^{\infty} 1/x dx$$

When we do the resulting improper integral, we get

$$\lim_{A \rightarrow \infty} \int_1^A 1/x dx = \lim_{A \rightarrow \infty} (\ln|A| - \ln|1|) = +\infty$$

We conclude from this that the harmonic series diverges.

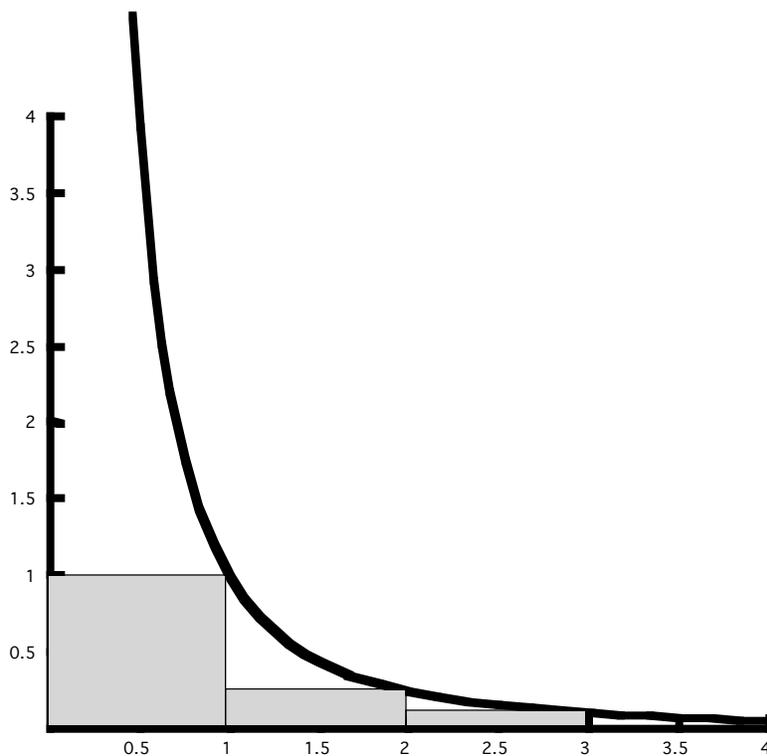
Here is another example. This time around, lets see if we can guess what might happen before we get into the details.

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

The first thing you should do should be to write a simple integral that looks like the series without thinking too much about the picky details such as the limits of integration.

$$\int_1^{\infty} \frac{1}{x^2} dx = 1$$

Given that we now suspect that the series may converge, we ought to try to get a set of rectangles that sit under the graph of $y = 1/x^2$. The rectangles depicted below show how to do this.



This time around the rectangles sit under the curve. We did this by giving each rectangle a height given by the value of the function at the right hand end-point of the interval it sits over. The picture suggests the following comparison.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \int_0^{\infty} \frac{1}{x^2} dx = +\infty$$

The comparison is not terribly useful, because the obvious integral to compare the series to diverges. This is easy to fix. All we have to do is to split off the first term of the series and use the integral comparison to control the other terms. That means using the portion of the curve over $[1, \infty)$ instead of $(0, \infty)$ for our comparison.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \int_1^{\infty} \frac{1}{x^2} dx = 1 + 1 = 2$$

Thus we see that the series converges.

Using an integral comparison to estimate the remainder in a series

Suppose we wanted to compute

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

We could estimate the value of this series by computing partial sums.

$$s_n = \sum_{k=1}^n \frac{1}{k^2}$$

Here is a table of partial sums for various values of N .

n	s_n
2	1.25
4	1.4236111111111112
8	1.527422052154195
16	1.5843465334449871
32	1.6141672628279242
64	1.6294305014088875

This gives us some sort of sense for what the sum will eventually be, but it does not tell us how many terms we need to add up in order to get the answer correct to within a certain predetermined accuracy. To understand how many terms we will need to add up in order to get that accuracy, it helps to write the series as a sum of two parts.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^n \frac{1}{k^2} + \sum_{k=n+1}^{\infty} \frac{1}{k^2}$$

$$\text{actual answer} = \text{part we can compute} + \text{error} \tag{1}$$

How do we get a sense for how big the error term is? The answer is that we can use some sort of integral comparison to control it.

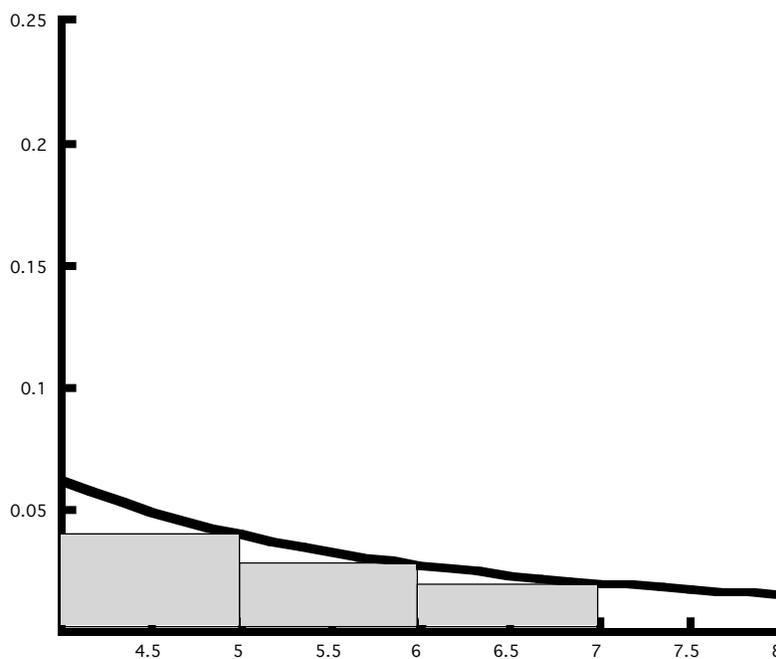
$$\int_{?}^{\infty} \frac{1}{x^2} dx < \sum_{k=n+1}^{\infty} \frac{1}{k^2} < \int_{?}^{\infty} \frac{1}{x^2} dx$$

We want to control the error term both from below and from above by means of some integral comparisons. It is clear what form those integrals have to take, so the only detail that remains to be determined is what bounds we need to set for the integrals to guarantee the comparisons we need. To see what the bounds should be, let us get a little more concrete by assuming a definite value of n .

$$n = 4$$

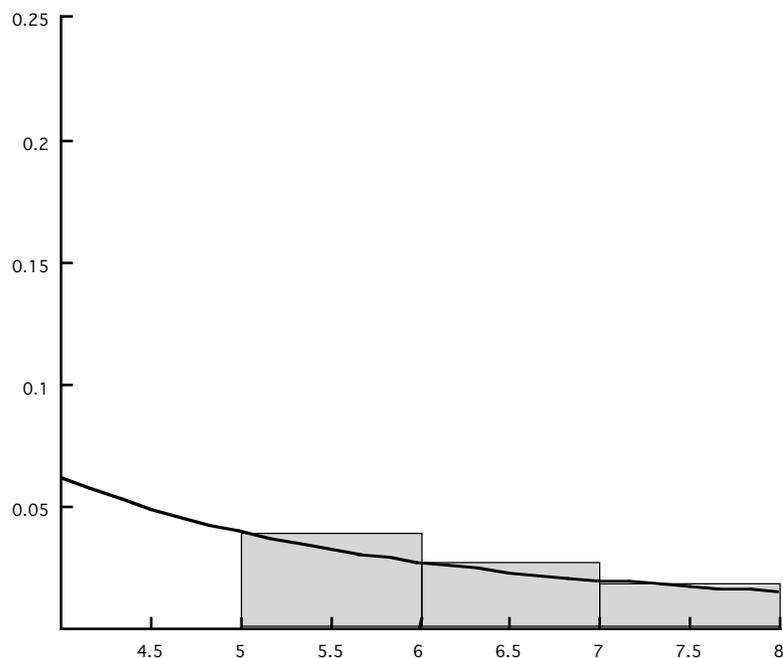
The first picture below shows how to set up a collection of rectangles to guarantee that

$$\sum_{k=5}^{\infty} \frac{1}{k^2} < \int_4^{\infty} \frac{1}{x^2} dx$$



Shifting those rectangles to the right by one unit reverses the inequality and also slightly changes the limits of integration used for the comparison.

$$\int_5^{\infty} \frac{1}{x^2} dx < \sum_{k=5}^{\infty} \frac{1}{k^2}$$



Putting this all together we have

$$\int_5^{\infty} \frac{1}{x^2} dx < \sum_{k=5}^{\infty} \frac{1}{k^2} < \int_4^{\infty} \frac{1}{x^2} dx$$

Computing the integrals we get

$$\frac{1}{5} < \sum_{k=5}^{\infty} \frac{1}{k^2} < \frac{1}{4}$$

Thus, for the special case $n = 4$ we have that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^4 \frac{1}{k^2} + \sum_{k=5}^{\infty} \frac{1}{k^2}$$

and the error term is no smaller than $1/5$ and no larger than $1/4$. Once we have seen how to do all of this for a specific value of n , the comparisons are easy to generalize to arbitrary n .

$$\frac{1}{n+1} = \int_{n+1}^{\infty} \frac{1}{x^2} dx < \sum_{k=n+1}^{\infty} \frac{1}{k^2} < \int_n^{\infty} \frac{1}{x^2} dx = \frac{1}{n}$$

For example, with $n = 64$ we have from the table above

$$S_{64} = 1.6294305014088875$$

and the estimate is accurate to

$$\frac{1}{65} < \text{Error with 64 terms} < \frac{1}{64}$$

Suppose I told you to get the answer accurate to $1/1000$. How many terms would you need to add? Well, the error you would make in adding up just n terms is

$$\frac{1}{n+1} < \sum_{k=n+1}^{\infty} \frac{1}{k^2} < \frac{1}{n}$$

If you want this error to be smaller than $1/1000$, you have to take n to be 1000 or more.