Final Exam Study Guide for Calculus III

Vector Algebra

1. The length of a vector and the relationship to distances between points
2. Addition, subtraction, and scalar multiplication of vectors, together with the geometric interpretations of these operations
3. Basic properties of vector operations (p.774)
4. The dot product – definition and basic properties (p.779)
5. The *geometric meaning* of the dot product in terms of lengths and angles – in particular the formula $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$
6. Vector projections – geometric meaning and formulas
7. The cross product – definition and basic properties (#1-4 of Theorem 8, p. 790)
8. The *geometric meaning* of the cross product – in particular $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$, with magnitude $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$, and direction given by the right-hand rule
9. $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.
10. $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is the volume of the parallelopiped spanned by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.
11. Cheap algebraic tests for geometric properties:
    - $\mathbf{a}$ and $\mathbf{b}$ are orthogonal $\Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$
    - $\mathbf{a}$ and $\mathbf{b}$ are parallel $\Leftrightarrow \mathbf{a} \times \mathbf{b} = 0$
    - $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar $\Leftrightarrow \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$

Lines and Planes

1. Intrinsic description (vectors) vs. Extrinsic description (scalar equations)

2. Lines: passage between a vector equation, parametric equations, and symmetric equations

3. Planes: passage between a vector description (a point together with two direction vectors) and a scalar equation

4. Using vector algebra to solve geometric problems about lines and planes – it is essential that you think geometrically and try to save the number crunching in components for the last moment.

Calculus of functions $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$

1. You should be able to state and motivate the following definitions using only vector language. But you should also know the convenient computational fact that everything can be checked or computed by working with component functions.

   - Limits and Continuity
   - Derivative of $\mathbf{r}(t)$ at $a \in \mathbb{R}$
   - Definite integral of $\mathbf{r}(t)$ on $[a, b]$

2. Differentiation rules (p. 826)

3. FTC I and II for continuous $\mathbf{r} : [a, b] \to \mathbb{R}^n$

4. The tangent line to the curve traced out by $\mathbf{r}(t)$ at a point $\mathbf{r}(t_0)$ – definition, geometric interpretation, and ability to find equations describing it (vector, parametric, or symmetric)

5. Interpretation of a continuous $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ as describing the motion of a particle in space, in which case $\mathbf{r}' =$velocity and $\mathbf{r}'' =$acceleration

Differential Calculus of functions $f : \mathbb{R}^n \to \mathbb{R}$

Definitions:

1. The level sets of a function f
2. The partial derivatives of f at a point $\mathbf{x}$
3. The directional derivative of a function f at a point $\mathbf{x}$ in the direction of a unit vector $\mathbf{u}$ [notation: $D_{\mathbf{u}}f(\mathbf{x})$]
4. The gradient of a function f at a point $\mathbf{x}$ [notation: $\nabla f(\mathbf{x})$]
5. Local and global extrema of f, and the difference between these two notions
6. The critical points of f

Results:

1. The Chain Rule
2. For f differentiable, the gradient determines all directional derivatives:
$$D_{\mathbf{u}}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$$
3. For f differentiable, the gradient points in the direction of greatest increase and has magnitude equal to that greatest rate of increase
4. For f differentiable, the gradient is orthogonal to the level sets of f
5. Multivariable Fermat's Theorem: If $\mathbf{x}$ is a local extremum for f, then $\mathbf{x}$ is a critical point for f
6. Clairaut's Theorem concerning the equality of mixed partial derivatives
7. Second Derivative Test for $f : \mathbb{R}^2 \to \mathbb{R}$

Proofs: You should be able to give careful and complete proofs of results 2, 3, and 5 above.

Techniques:

1. Using the geometry of the gradient and vector algebra to efficiently work with tangent planes to surfaces and tangent lines to curves

2. Closed and Bounded Set Method for optimizing functions of several variables

3. Lagrange Multipliers for optimizing a function of several variables subject to constraints

Integral Calculus of functions $f : \mathbb{R}^2 \to \mathbb{R}$

1. Definition of the double integral and motivation in terms of the volume problem

   a) rectangular regions: $\iint_R f(x,y)dA$
   b) general regions: $\iint_D f(x,y)dA$

2. Basic properties of the double integral (p. 958)

3. Fubini's Theorem and the computation of double integrals via iterated integrals

4. Using polar coordinates to compute double integrals

Calculus of Vector Fields $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$

1. Visualization of a vector field as a "field of arrows" and interpretation as a force field

2. Definition of a conservative vector field

3. Definition of curve / line integrals of scalar functions $f : \mathbb{R}^n \to \mathbb{R}$

   a) with respect to arclength: $\int_C f ds$
   b) with respect to x, y, z: $\int_C f dx$, etc.
   c) formulas for the computation of the integrals from a) and b) involving a parametrization $\mathbf{r} : [a,b] \to \mathbb{R}^n$ for C:

$$\int_C f ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt \qquad \int_C f dx = \int_a^b f(\mathbf{r}(t))x'(t)dt$$

4. Curve / line integrals of vector fields $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$

   a) Why are all of the following integrals equal?

   $$\int_C \mathbf{F}\cdot d\mathbf{r} := \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_C (\mathbf{F}\cdot\mathbf{T})ds = \int_C Pdx+Qdy+Rdz$$

   (Here $\mathbf{F} = (P, Q, R)$ and $\mathbf{T}(t)$ = unit tangent vector to C at $\mathbf{r}(t)$.)

   b) Motivation for the definition of the integral from a) in terms of the work done by the force field $\mathbf{F}$ in moving a particle along C

5. Statement and proof of the Conservative Vector Field Theorem

6. Theorems characterizing conservative vector fields:

   a) $\mathbf{F} : D \to \mathbf{R}^n$ is conservative if and only if $\int_C \mathbf{F}\cdot d\mathbf{r}$ is path independent in D (here D is open and connected, and $\mathbf{F}$ is continuous)

   b) $\mathbf{F} = (P, Q) : D \to \mathbf{R}^2$ is conservative if and only if $P_y = Q_x$ on D (here D is simply connected and P, Q have continuous first partial derivatives)

7. Finding potential functions for conservative vector fields via partial integration

8. Green's Theorem