NOTES ON COMPLEX ANALYSIS

ALAN PARKS, LAWRENCE UNIVERSITY

Contents

These notes are meant to provide an outline of our course (math 535, Spring 2016) and to supplement our primary text: *A First Course in Complex Analysis* by Beck, Marchesi, Pixton, Sabalka.

`http://math.sfsu.edu/beck/papers/complex.pdf`

Please read the copyright restrictions given on p.2 of the pdf.

We begin with properties of the real numbers that are normally studied in a first course in advanced calculus or real analysis, such as *Foundations of Analysis* (math 310).

1. Analysis Basics

Completeness of the Reals. *Every non-empty subset of the reals that is bounded above has a least upper bound (its* sup*); every non-empty subset of the reals that is bounded below has a greatest lower bound (its* inf*).*

If x is the sup of S, and if $\epsilon > 0$, there is $y \in S$ with $x - \epsilon < y \leq x$. If x is the inf of S, and if $\epsilon > 0$, there is $y \in S$ with $x \leq y < x + \epsilon$.

Date: Spring Term, 2016.
These notes were begun in May 1996 and have been revised and extended periodically in the interim.

Monotone Convergence. *Let a_n define a real sequence. If $a_n \leq a_{n+1}$ for all n (if a_n is* increasing*), and if a_n is bounded above, then $a_n \to A$ as $n \to \infty$, where A is the sup of the set of all a_n. If $a_n \geq a_{n+1}$ for all n (if a_n is* decreasing*), and if a_n is bounded below, then $a_n \to A$ as $n \to \infty$, where A is the inf of the set of all a_n.*

We need you to recall the *length* or *norm* of a point or vector. If $a, b \in \mathbb{R}$, then

$$|(a,b)| = \sqrt{a^2+b^2}$$

In complex analysis, we use the word *modulus* for this number. The modulus of (a,b) is its distance from $(0,0)$, and the geometric distance between points p, q is $|p-q|$. You are supposed to know the Triangle Inequality:

$$|p+q| \leq |p| + |q| \quad \text{for all} \quad p, q \in \mathbb{R}^2$$

Given $w \in \mathbb{R}^2$ and $r > 0$, we define $D(w;r)$ to be the set of points $z \in \mathbb{R}^2$ such that $|z-w| < r$. The set $D(w;r)$ is called the *open disk centered at w of radius r*. It is easy to see that $D(w;r)$ is the interior of the circle centered at w of radius r. Open disks will serve as the natural domain for many facts in our course.

Here is a fact about partial derivatives from Calculus III.

Proposition 1. *Let $(a,b) \in \mathbb{R}^2$, let $r > 0$, and define $D = D((a,b);r)$. Suppose that $u : D \to \mathbb{R}$ has first partial derivatives at every point in the disk. Let $(c,d) \in D$. Then there are real numbers a_1, b_1 such that a_1 is between a and c, and b_1 is between b and d, and*

$$u(c,d) - u(a,b) = \frac{\partial u}{\partial x}(a_1, b) \cdot (c-a) + \frac{\partial u}{\partial y}(c, b_1) \cdot (d-b)$$

Proof. Write

$$u(c,d) - u(a,b) = u(c,d) - u(c,b) + u(c,b) - u(a,b)$$

Since $\partial u/\partial y$ exists, the Mean Value Theorem can be applied to $u(c,d) - u(c,b)$ to find b_1 between b and d such that

$$u(c,d) - u(c,b) = \frac{\partial u}{\partial y}(c, b_1) \cdot (d-b)$$

Similarly, the Mean Value Theorem can be applied to $u(c,b) - u(a,b)$ to find a_1 between a and c such that

$$u(c,b) - u(a,b) = \frac{\partial u}{\partial x}(a_1, b) \cdot (c-a)$$

□

We will write the conclusion of Proposition 1 as a statement about Δu in terms of Δx and Δy:

$$\Delta u = \frac{\partial u}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial y} \cdot \Delta y \tag{1}$$

We have to remember all the actual conditions given in Proposition 1, but we will find that equation (1) provides a very useful shorthand.

2. The Complex Plane

Definition and Arithmetic

One of the simplest and most useful ways to define the complex numbers is to declare them to be the points in the plane $\mathbb{R}^2$. We write $\mathbb{C}$ for the set of complex numbers, and we are saying that $\mathbb{C} = \mathbb{R}^2$. For a complex number (x, y), we call x the *real part* and y the *imaginary part*. We write

$$x = \mathrm{Re}(x, y) \quad \text{and} \quad y = \mathrm{Im}(x, y)$$

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

We will regard the real numbers to be a subset of the complex numbers, identifying the real number x with the point $(x, 0)$. We do not define comparision of complex numbers in general, and so when we write an inequality such as $x > 0$, we are implying that x is a real number. Similarly, an inequality such as $r \leq 2$ would indicate that r is real, as well.

Here is the how addition and multiplication are defined:

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b) \cdot (c, d) = (a \cdot c - b \cdot d, a \cdot d + b \cdot c)$$

You should notice that $(x, 0)+(y, 0) = (x+y, 0)$ and $(x, 0)\cdot(y, 0) = (xy, 0)$, according with our identification of real numbers with points on the x-axis. Also notice that

$$(c, 0) \cdot (a, b) = (ca, cb) = c \cdot (a, b)$$

so that complex multiplication by the real number c is the same as scalar multiplication in $\mathbb{R}^2$.

The operations just defined make $\mathbb{C}$ into a *field* – that means that the addition and multiplication are associative and commutative, that multiplication distributes over addition, that there is an additive identity element, that there is a multiplicative identity element, that every element has an additive inverse, and that every non-zero element has a multiplicative inverse. The additive identity is, of course, $0 = (0, 0)$, and the multiplicative identity is $1 = (1, 0)$. Here are the multiplicative inverses: let $(a, b) \neq (0, 0)$, so that $a^2 + b^2 \neq 0$ as real numbers, and compute that

$$(a, b) \cdot \left(\frac{a}{a^2 + b^2} \, , \, -\frac{b}{a^2 + b^2} \right) = (1, 0)$$

Define $i = (0, 1)$, and notice that

$$i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1$$

This equation identifies i as the expected *imaginary* number, although we hope you don't think that the point $(0, 1)$ is imaginary!

We can interpret $(a, b) \in \mathbb{C}$ in several ways. Notice that

$$(a, b) = (a, 0) + b \cdot (0, 1) = a + b \cdot i$$

The expression $a+b\cdot i$ gives a very common representation of complex numbers; this representation is called the *rectangular representation*. Recall that a is the real part and b is the imaginary part of $a + b \cdot i$. Because multiplication is commutative, it doesn't matter whether $a + b \cdot i$ is written $a + i \cdot b$; each is used.

We are also interested in a *polar representation.* For a non-zero complex number z, we consider polar coordinates (r, θ), where

$$r > 0 \quad \text{and} \quad z = r \cdot \cos(\theta) + i \cdot r \cdot \sin(\theta)$$

If we write the rectangular representation, $z = x + i \cdot y$, then

$$x^2 + y^2 = r^2 \cdot \cos^2(\theta) + r^2 \cdot \sin^2(\theta) = r^2$$

and since $r > 0$, we see that $r = \sqrt{x^2 + y^2}$, the previously mentioned *modulus* of z, denoted $|z|$.

Because of the periodicity of cosine and sine, there are many possible angles θ that can be used in the polar representation. We call θ an *argument* of z; the other possible arguments are the numbers $\theta + 2 \cdot \pi \cdot k$, where k is an integer.[1] The set of all arguments of z is denoted $\mathrm{Arg}(z)$.

It will be convenient to avoid polar representations of the complex number 0; although it has modulus 0, but its set of arguments would be the entire set of real angles – allowing that case would be an inconvenience.

Later, we will see that

$$\cos(\theta) + i \cdot \sin(\theta) = \exp(i \cdot \theta) \tag{2}$$

(often written $e^{i \cdot \theta}$) We will give a formal construction of the exponential later. For now, we assume familiarity with the real exponential function e^x (for $x \in \mathbb{R}$), and we will follow custom and use $e^{i \cdot \theta}$ for (2). Notice that

$$\cos(0) + i \cdot \sin(0) = 1 + i \cdot 0 = 1$$

which agrees with $\exp(i \cdot 0) = \exp(0) = 1$. Second, the angle addition formulas show, for all $\alpha, \beta \in \mathbb{R}$, that

$$\big[\cos(\alpha) + i \cdot \sin(\alpha)\big] \cdot \big[\cos(\beta) + i \cdot \sin(\beta)\big] = \cos(\alpha + \beta) + i \cdot \sin(\alpha + \beta) \tag{3}$$

and this tells us that

$$\exp(i \cdot \alpha) \cdot \exp(i \cdot \beta) = \exp(i \cdot (\alpha + \beta))$$

which looks like a familiar identity for the exponential function. For now, we will let the formula (2) *define* $\exp(i \cdot \theta)$.

Thus, polar representation of $z \neq 0$ has the form

$$z = |z| \cdot \exp(i \cdot \theta) \quad \text{where} \quad \theta \in \mathrm{Arg}(z)$$

This makes it clear that $|z| = 1$ if and only if $z = \exp(i \cdot \theta)$ for some θ.

Proposition 2. *Let $z, w \in \mathbb{C}$. Then $|z \cdot w| = |z| \cdot |w|$.*

Proof. The equation is obvious if $z = 0$ or if $w = 0$. Otherwise, write the polar representation of each:

$$z = |z| \cdot \exp(i \cdot \alpha) \quad \text{and} \quad w = |w| \cdot \exp(i \cdot \beta)$$

so that

$$z \cdot w = |z| \cdot |w| \cdot \exp(i \cdot (\alpha + \beta))$$

We have $|z| \cdot |w| > 0$, and so it is the modulus of $z \cdot w$. $\square$

[1] It goes without saying that we measure angles in radians.

DeMoivre's Theorem shows how to find n-th roots. In the following, $\sqrt[n]{r}$ is the ordinary real n-th root: the unique positive real number whose n-th power is r.

DeMoivre's Theorem. *Let z be a non-zero complex number, and let $\theta \in Arg(z)$. Let n be a positive integer. Then there are exactly n complex numbers w with $w^n = z$. Each such w has the form*

$$w = \sqrt[n]{|z|} \cdot \exp((\theta + 2\pi \cdot k)/n) \quad \text{where} \quad k = 0, 1, \ldots, n-1$$

Conjugates

The *complex conjugate*[2] of the complex number $x + i \cdot y$, where $x, y \in \mathbb{R}$, is

$$\overline{x + i \cdot y} = x - i \cdot y \quad \text{which is to say} \quad \overline{(x, y)} = (x, -y)$$

In other words, $\mathrm{Re}\overline{z} = \mathrm{Re}z$ and $\mathrm{Im}\overline{z} = -\mathrm{Im}z$.

There is an important geometric interpretation to the conjugate: it maps the point (x, y) to its reflection across the x-axis.

What does this look like in the polar representation? We need to remember that

$$\cos(-\theta) = \cos(\theta) \quad \text{and} \quad \sin(-\theta) = -\sin(\theta) \quad \text{for all} \quad \theta \in \mathbb{R}$$

Thus, if $r > 0$ and $\theta \in \mathbb{R}$, then

$$\begin{aligned}\overline{r \cdot \exp(i \cdot \theta)} &= \overline{r \cdot \cos(\theta) + i \cdot r \cdot \sin(\theta)} = r \cdot \cos(\theta) - i \cdot r \cdot \sin(\theta) \\ &= r \cdot \cos(-\theta) + i \cdot r \cdot \sin(-\theta) = r \cdot \exp(-i \cdot \theta)\end{aligned}$$

An important corollary: the complex conjugate of $\exp(i \cdot \alpha)$ is its multiplicative inverse $\exp(-i \cdot \alpha)$.

Here are the properties of the conjugate. We will leave the proof to you or to class. Proposition 3(a,d) are best done using the rectangular representation, Proposition 3(b,c), the polar.

Proposition 3. *Let $z, w \in \mathbb{C}$. Then*
(a) $\overline{z + w} = \overline{z} + \overline{w}$
(b) $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$
(c) $z \cdot \overline{z} = |z|^2$
(d) $z = \overline{z}$ *if and only if* $z \in \mathbb{R}$

Topology

We have already defined the open disk $D(w; r)$:

$$D(w; r) = \left\{ z \in \mathbb{C} \;\middle|\; |z - w| < r \right\}$$

A subset A of the complex numbers is *bounded* if it is contained in some open disk. If we wish, we can always assume that 0 is the center of disk. Indeed, suppose that $A \subseteq D(w; r)$ for some $w \in \mathbb{C}$ and $r > 0$. If $a \in A$, then the Triangle Inequality shows that

$$|a| = |a - w + w| \leq |a - w| + |w| < r + |w|$$

and so $A \subseteq D(0; r + |w|)$.

---

[2]The word *conjugate* is used in several contexts in mathematics. For our course, it will always refer to the complex conjugate, and so we will usually leave off the word *complex*.

A set $V \subseteq \mathbb{C}$ is *open in* $\mathbb{C}$ if for every $z \in V$, there is $r > 0$ with $D(z; r) \subseteq V$. We will see that open sets are the natural domain of the functions we will study in the course. Here are the basic properties of open sets.

Proposition 4. *The set $\mathbb{C}$ and the empty set ϕ are open in $\mathbb{C}$. Every open disk is open in $\mathbb{C}$. If U, V are open subsets of $\mathbb{C}$, then so is $U \cap V$. If $U_1, U_2, \ldots$ are open subsets of $\mathbb{C}$, then so is their union $\bigcup_{j=1}^{\infty} U_j$.*

Proof. We will prove only that open disks are open and leave the other arguments to you or to class. Let $w \in D(z; r)$. Then $|w - z| < r$, define $s = r - |w - z|$ so that $s > 0$. We claim that $D(w; s) \subseteq D(z; r)$. Indeed, let $u \in D(w; s)$ and use the Triangle Inequality:

$$|u - z| = |u - w + w - z| \leq |u - w| + |w - z| < s + |w - z| = r$$

This proves that $u \in D(z; r)$. □

It will often be convenient to work with the complement of an open set. The set $C \subseteq \mathbb{C}$ is *closed in* $\mathbb{C}$ if the set $\mathbb{C} \setminus C$ is open. To repeat: C is closed if, for all $z \in \mathbb{C}$ with $z \notin C$, there is $r > 0$ such that $D(z; r) \cap C = \phi$. Here is a direct analog of Proposition 4.

Proposition 5. *The set ϕ and the set $\mathbb{C}$ are closed in $\mathbb{C}$. If C, E are closed subsets of $\mathbb{C}$, then so is $C \cup E$. If $C_1, C_2, \ldots$ are closed subsets of $\mathbb{C}$, then so is their intersection $\bigcap_{j=1}^{\infty} C_j$.*

We will sometimes want to include the circular boundary of a disk. For $z \in \mathbb{C}$ and $r > 0$, define

$$\overline{D}(z; r) = \{ w \in \mathbb{C} \mid |w - z| \leq r \}$$

Thus, $D(z; r) \subset \overline{D}(z; r)$ with $\overline{D}(z; r) \setminus D(z; r)$ being the circle of points $w \in \mathbb{C}$ such that $|w - z| = r$. The set $\overline{D}(z; r)$ is called a *closed disk*; it is a closed set.

We turn to properties of sequences. Let $a_n \in \mathbb{C}$ for $n = 0, 1, 2, \ldots$; then we call a_n a *complex sequence.* A sequence is *bounded* if its set of values is bounded. The following fact generalizes the Bolzano-Weierstrass Theorem in the reals.

Proposition 6. *Let a_n be a bounded complex sequence. Then there is $c \in \mathbb{C}$ such that for all $\epsilon > 0$, there are infinitely many positive integers n such that $|a_n - c| < \epsilon$.*

A corollary we will need later:

Proposition 7. *Let C_n be non-empty, closed subsets of $\mathbb{C}$, for $n = 1, 2, \ldots$, and suppose that $C_{n+1} \subseteq C_n$ for each n. Suppose that C_1 is bounded. Then there is a complex number in the intersection of all the C_n.*

Proof. Each C_n is non-empty: choose $c_n \in C_n$ for each n. We have $C_n \subseteq C_1$ for every n, and the set C_1 is bounded; this shows that c_n is a bounded sequence. By Proposition 6 there is a complex number c such that $|c_n - c| < \epsilon$ for infinitely many n, for each ϵ.

We claim that c is in all the C_n. If not, let $c \notin C_k$. Since C_k is closed, there is $\delta > 0$ such that $D(c; \delta) \cap C_k = \phi$. There are infinitely many n such that $c_n \in D(c; \delta)$, and so there is such an $n \geq k$. We have $c_n \in C_n \subseteq C_k$, and so $c_n \in C_k \cap D(c; \delta)$, a contradiction. □

3. Limits and Continuity

The definitions are exactly as in the real case, given that the complex modulus replaces the real absolute value. Let $A \subseteq \mathbb{C}$. Then $a \in \mathbb{C}$ is a *limit point*[3] of A if for all $\delta > 0$, the set $D(a;\delta) \cap A$ has at least two elements. It follows that $D(a;\delta) \cap A$ has infinitely many elements.

Here is the function limit. Let $A \subseteq \mathbb{C}$ and let $f : A \to \mathbb{C}$. Let a be a limit point of A, and let $L \in \mathbb{C}$. Then

$$\lim_{z \to a} f(z) = L$$

means this: for all $\epsilon > 0$ there is $\delta > 0$ such that if $b \in A$ and $0 < |b - a| < \delta$, then $|f(b) - L| < \epsilon$.

We get the expected uniqueness of the limit,[4] if it exists, and we get the limit algebra over sums, constant multiples, products, and ratios (when the denominator goes to a non-zero number). The ordinary limit does not behave well in composite functions, in general.

In the real case, $x \to \infty$ makes sense, because there is only one way to approach positive infinity. In the complex case, it is not clear what ∞ means, since, in its widest interpretation it can be approached in any direction from the origin. It usually makes more sense to speak of $|z| \to \infty$ in the complex case. Here is a formal definition: Let $A \subseteq \mathbb{C}$, where A is unbounded, and let $f : A \to \mathbb{C}$. Let $L \in \mathbb{C}$, and then

$$\lim_{|z| \to \infty} f(z) = L$$

means that for all $\epsilon > 0$, there is a real number M such that if $|z| > M$ and $z \in A$, then $|f(z) - L| < \epsilon$. The simplest example: $1/z \to 0$ as $|z| \to \infty$.

We also have limits of the modulus going to infinity, with the usual definition. Here is an example: a typical, useful fact about polynomials similar to the real case.

Proposition 8. *Let $f(z)$ be a polynomial with complex coefficients and of degree at least 1. Then we have the following limit in the complex numbers:*

$$\lim_{|z| \to \infty} |f(z)| = \infty$$

Proof. Write

$$f(z) = \sum_{k=0}^{n} a_k \cdot z^k$$

where $a_k \in \mathbb{C}$ for each k, and $a_n \neq 0$, with $n \geq 1$. Choose a positive real number $\delta \geq 1$ with

$$\delta > \frac{1}{|a_n|} \cdot \sum_{k=0}^{n-1} |a_k| \quad \text{and then} \quad |a_n| - \frac{1}{\delta} \cdot \sum_{k=0}^{n-1} |a_k| > 0$$

Denote by b the number on the left of the second inequality.

Let $|z| \geq \delta$, and since $\delta \geq 1$, we have $|z^k| \geq |z| \geq \delta$ for all $k \geq 1$.

[3]Some use the term *accumulation point*.

[4]Uniqueness is why the number a needs to be a limit point of A.

Now use the (reverse) Triangle Inequality to estimate

$$\begin{aligned}
|f(z)| &= \left|\sum_{k=0}^{n} a_k \cdot z^k\right| = |z^n| \cdot \left|a_n + \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right| \\
&\geq |z|^n \cdot \left[|a_n| - \left|\sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right|\right] \geq |z|^n \cdot \left[|a_n| - \sum_{k=0}^{n-1} \left|\frac{a_k}{z^{n-k}}\right|\right] \\
&= |z|^n \cdot \left[|a_n| - \sum_{k=0}^{n-1} \frac{|a_k|}{|z^{n-k}|}\right] \geq |z|^n \cdot \left[|a_n| - \sum_{k=0}^{n-1} \frac{|a_k|}{\delta}\right] \\
&= |z|^n \cdot b
\end{aligned}$$

Since $n \geq 1$ and $b > 0$, we see that $|f(z)| \to \infty$ as $|z| \to \infty$. □

The definition of *continuity* in complex functions is identical to that for the real case, using the modulus. Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{C}$. Let $a \in A$. Then f is *continuous at a* if for all $\epsilon > 0$, there's $\delta > 0$ such that if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

As in the real case, it doesn't matter whether $a \in A$ is a limit point, or not, but if it is, then f being continuous at a is exactly that $f(z) \to f(a)$ as $z \to a$.

The function $f : A \to \mathbb{C}$ is *continuous* (or *continuous on A*) if it is continuous at all the elements of A.

The sum, difference, and product of functions continuous on $A \subseteq \mathbb{C}$ is continuous on A. Constant multiples of continuous functions are continuous, as well. If f, g are continuous on A, and if $g(z) \neq 0$ for all $z \in A$, then $f(z)/g(z)$ is continuous on A.

It follows that every polynomial is continuous on $\mathbb{C}$. Every *rational function* (ratio of polynomials) is continuous wherever the denominator is not 0. Recall that a polynomial of degree n can have at most n roots. Thus, every rational function is continuous on the open set obtained by removing the finitely many roots of the denominator – its domain is an open subset of the complex numbers.

Composites: if $A, B \subseteq \mathbb{C}$, if $f : A \to B$ is continuous, and if $g : B \to \mathbb{C}$ is continuous, then $g(f(z))$ is continuous on A. This makes continuity a much more useful concept than the mere limit.

Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{C}$. For each $a \in A$, we can write $f(a) = u(a) + i \cdot v(a)$ in rectangular representation, where $u, v : A \to \mathbb{R}$. We can also write $a = x + i \cdot y$ in rectangular, and this allows us to think of $u(x, y)$ and $v(x, y)$ as real functions of two real variables. We say that u is the *real part of f* and v is the *imaginary part of f*. We will prove the following in class or as a homework exercise.

Proposition 9. *Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{C}$. Let u, v be the real and imaginary parts of f, respectively. Then f is continuous if and only if u and v are continuous (as real functions of two variables).*

We need an additional fact.

Proposition 10. *Let $C \subset \mathbb{C}$ be closed and bounded. Let $f : C \to \mathbb{C}$ be continuous. Then $f(C)$ is bounded.*[5]

[5]It is the case that $f(C)$ is closed, as well, but we will not need that fact.

Proof. Assume that $f(C)$ is not bounded. Then, for each positive integer n there is $a_n \in C$ such that $|f(a_n)| \geq n$. Proposition 6 finds a number c such that, for all $\delta > 0$, there are infinitely many n such that $a_n \in D(c;\delta)$. Because C is closed, we have $c \in C$.

Since f is continuous at c, there is $\delta > 0$ such that if $|z-c| < \delta$ and $z \in C$, then $|f(z) - f(c)| < 1$. Among the infinitely many n such that $a_n \in D(c;\delta)$, choose such an n with $n > |f(c)| + 1$. Then

$$\begin{aligned} |f(a_n)| &\leq |f(a_n) - f(c)| + |f(c)| \\ &< 1 + |f(c)| < n \end{aligned}$$

contradicting the fact that $|f(a_n)| \geq n$. □

Let $A \subseteq \mathbb{C}$ and let $f_n : A \to \mathbb{C}$ for $n = 0, 1, 2, \ldots$. Then $f_n \to f$ *uniformly* means that for all $\epsilon > 0$, there is N such that if $n \geq N$ and $x \in A$, then $|f_n(x) - f(x)| < \epsilon$. The reader should remember that the N is chosen before x is specified.

Proposition 11. *Let $A \subseteq \mathbb{C}$ and let $f_n : A \to \mathbb{C}$ be continuous for $n = 0, 1, 2, \ldots$. Suppose that $f_n \to f$ uniformly. Then f is continuous.*

Proof. Let $a \in A$ and $\epsilon > 0$. Get N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$. Choose $n \geq N$. Since $f_n(x)$ is continuous, there is $\delta > 0$ such that if $|x - a| < \delta$, then $|f_n(x) - f_n(a)| < \epsilon$. For such x, compute

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| < 3 \cdot \epsilon$$

This proves that $f(x)$ is continuous at a; since a is arbitrary, f is continuous. □

4. The Derivative

The definition of the derivative in the complex plane looks exactly like the definition over the reals. That similarity hides significant differences, as we will see. Let $V \subseteq \mathbb{C}$, and let $f : V \to \mathbb{C}$. Let $w \in V$ be a limit point of V. If the following limit exists:

$$\lim_{z \to w} \frac{f(z) - f(w)}{z - w} \tag{4}$$

then we write $f'(w)$ for the limit and call it the *derivative* of f at w. The notation $z \to w$ requires $z \in V$ and $z \neq w$.

To be absolutely clear, let's rehearse the meaning of the limit (4). The limit says that for each $\epsilon > 0$, there is $\delta > 0$ such that if $z \in V$ and $z \neq w$ and $|z - w| < \delta$, then

$$\left| \frac{f(z) - f(w)}{z - w} - f'(w) \right| < \epsilon$$

We can always replace a given δ by a smaller positive numbers. Since w is a limit point of V, there are always infintely many elements $z \in V$ with $|z - w| < \delta$. For that reason, the limit $f'(w)$ is unique, if it exists.

Observe that if $V \subseteq \mathbb{R}$ and $f : V \to \mathbb{R}$, then (4) gives the ordinary derivative of calculus. We are also interested in the case where $V = [a, b]$, a closed interval on the reals, and $f : V \to \mathbb{C}$. We call f a *curve* in this case, and $f'(t)$ is what might be called the *vector derivative* – if we write $f(t) = u(t) \cdot i \cdot v(t)$, where u, v are the real and imaginary parts, then

$$f'(t) = \frac{du}{dt} + i \cdot \frac{dv}{dt}$$

The following facts are immediate, using proofs very similar to those used in the real case. Let $V \subseteq \mathbb{C}$, let $f : V \to \mathbb{C}$ and $g : V \to \mathbb{C}$. Let $w \in V$ be a limit point of V, and suppose that $f'(w)$ and $g'(w)$ exist.

(1) f is continuous at w
(2) If $\alpha \in \mathbb{C}$, then $(\alpha \cdot f)'(w) = \alpha \cdot f'(w)$
(3) Then $(f + g)'(w) = f'(w) + g'(w)$.
(4) Then $(f \cdot g)'(w) = f'(w) \cdot g(w) + f(w) \cdot g'(w)$
(5) If $g(w) \neq 0$, then

$$\left(\frac{f}{g}\right)'(w) = \frac{f'(w) \cdot g(w) - f(w) \cdot g'(w)}{g(w)^2}$$

Our version of the Chain Rule is subtle in that it covers two rather different cases, as we will see.

Chain Rule. *Let $A \subseteq \mathbb{C}$ and let B be an open subset of $\mathbb{C}$. Let $f : A \to B$ and $g : B \to \mathbb{C}$. Let $w \in A$ be a limit point of A, and suppose that $f'(w)$ exists. Suppose that $g'(f(w))$ exists. Then $(g(f))'(w) = g'(f(w)) \cdot f'(w)$.*

Proof. Define $h : B \to \mathbb{C}$ by

$$h(z) = \begin{cases} \frac{g(z)-g(f(w))}{z-f(w)} & \text{if } z \neq f(w) \\ g'(f(w)) & \text{z=f(w)} \end{cases}$$

The fact that $h(z) \to g'(f(w))$ as $z \to f(w)$ (with $z \neq f(w)$) shows that h is continuous at $f(w)$. Observe that

$$h(z) \cdot (z - f(w)) = g(z) - g(f(w)) \quad \text{for all} \quad z \in B$$

for when $z = f(w)$ the equation says merely that $0 = 0$.

Let $p \in A \setminus \{w\}$ and compute

$$\frac{g(f(p) - g(f(w))}{p - w} = \frac{h(f(p)) \cdot (f(p) - f(w))}{p - w} = h(f(p)) \cdot \frac{f(p) - f(w)}{p - w}$$

As $p \to w$, the fraction on the right goes to $f'(w)$. Since $f'(w)$ exists, $f(p)$ is continuous at w, and so as $p \to w$, we have $f(p) \to f(w)$. Since h is continuous at w, we have $h(f(p)) \to h(f(w)) = g'(f(w))$. □

5. The Cauchy-Riemann Equations

Complex Analysis, as a subject, is primarily concerned with the following situation: Let V be an open subset[6] of the complex numbers and $f : V \to \mathbb{C}$. If $f'(w)$ exists at all points $w \in V$, we say that f is *holomorphic*[7] *on* V.

[6]Observe that all elements of an open subset are limit points of the subset.

[7]We will discuss the use of the word *holomorphic* in class. It will become apparent that it would not serve our purpose to refer to f as *differentiable*, as we do in the real case.

If A, B are open subsets of $\mathbb{C}$, if $f : A \to B$ and $g : B \to \mathbb{C}$, and if f and g are holomorphic on their domains, then the Chain Rule shows that $g(f(z))$ is holomorphic on A.

It will be useful to express the derivative of f in terms of partial derivatives in its real and imaginary parts. Let V be an open subset of the comples numbers, and let $f : V \to \mathbb{C}$. Write $f(z) = u(z) + i \cdot v(z)$, where u, v are the real and imaginary parts of $f(z)$, respectively. Recall that we think of $u(z)$ as $u(x, y)$, where $z = x + i \cdot y$.

The equations (5) in the following are the famous *Cauchy-Riemann* equations.

Proposition 12. *Let V be an open subset of $\mathbb{C}$ and let $f : V \to \mathbb{C}$. Write $f = u+i\cdot v$ in rectangular form. If $w \in V$, and if $f'(w)$ exists, then u, v have first partial derivatives at w, and*

$$f'(w) = \frac{\partial u}{\partial x}(w) + i \cdot \frac{\partial v}{\partial x}(w) = -i \cdot \frac{\partial u}{\partial y}(w) + \frac{\partial v}{\partial y}(w)$$

and so at w, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{5}$$

Proof. Write $w = a + i \cdot b$, its rectangular representation. We let $z \to w$ by setting $z = x + i \cdot b$ and having $x \to a$. Then

$$\begin{aligned} f'(w) = \lim_{z\to w} \frac{f(z) - f(w)}{z - w} &= \lim_{x\to a} \frac{u(x, b) - u(a, b) + i \cdot (v(x, b) - v(a, b))}{x - a} \\ &= \lim_{x\to a} \frac{u(x, b) - u(a, b)}{x - a} + i \cdot \lim_{x\to a} \frac{v(x, b) - v(a, b)}{x - a} \\ &= \frac{\partial u}{\partial x}(w) + i \cdot \frac{\partial v}{\partial x}(w) \end{aligned}$$

We see that the partial derivatives of u and v with respect to x exist at w.

We repeat the limit, this time letting $z = a + i \cdot y$ and having $y \to b$. We get

$$\begin{aligned} f'(w) = \lim_{z\to w} \frac{f(z) - f(w)}{z - w} &= \lim_{y\to b} \frac{u(a, y) - u(a, b) + i \cdot (v(a, y) - v(a, b))}{i \cdot (y - b)} \\ &= -i \cdot \lim_{y\to b} \frac{u(a, y) - u(a, b)}{y - b} + \lim_{y\to b} \frac{v(x, b) - v(a, b)}{y - b} \\ &= -i \cdot \frac{\partial u}{\partial y}(w) + \frac{\partial v}{\partial y}(w) \end{aligned}$$

Equating the real parts and imaginary parts in the two expressions for the derivatives, we obtain (5). $\square$

Here is a consequence of the Cauchy-Riemann equations: a function with zero derivative in a disk is constant there. This generalizes a fact from real analysis.

Proposition 13. *Let $c \in \mathbb{C}$ and $r > 0$ and suppose that $f(z)$ has derivative 0 at every point of $D(c; r)$. Then $f(z)$ is constant on the disk.*

Proof. From the equations for $f'(z)$, we see that the first partial derivatives of its real and imaginary parts of f are 0 in the disk. Proposition 1 then shows that the real and imaginary parts of f are constant. $\square$

We will see that the converse of Proposition 12 is true – that is a major theorem. For now, we can prove a partial converse – partial in the sense that we add a hypothesis.

Proposition 14. *Let V be an open subset of $\mathbb{C}$, and let $u, v : V \to \mathbb{R}$ have continuous first partial derivatives and also satisfy the Cauchy-Riemann equations (5). Then $f(z) = u(z) + i \cdot v(z)$ defines a holomorphic function on V.*

Proof. Let $w \in V$ and get $r > 0$ such that $D(w;r) \subseteq V$. For $z \in D(w;r)$ with $z \neq w$, write $\Delta z = z - w$, and write $\Delta z = \Delta x + i \cdot \Delta y$ to give its rectangular representation.

We apply Proposition 1 (using the equation form (1)) to both u and v, and we have[8]

$$\begin{aligned}\Delta u = \Delta_x u + \Delta_y u &= \frac{\partial u}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial y} \cdot \Delta y \\ &= \frac{\partial u}{\partial x} \cdot \Delta x - i \cdot \frac{\partial u}{\partial y} \cdot (i \cdot \Delta y) = \frac{\partial u}{\partial x} \cdot \Delta x + i \cdot \frac{\partial v}{\partial x} \cdot (i \cdot \Delta y)\end{aligned}$$

using Cauchy-Riemann for the last equation. Also

$$\Delta v = \Delta_x v + \Delta_y v = \frac{\partial v}{\partial x} \cdot \Delta x + \frac{\partial v}{\partial y} \cdot \Delta y \qquad \text{so that}$$

$$i \cdot \Delta v = i \cdot \frac{\partial v}{\partial x} \cdot \Delta x + i \cdot \frac{\partial v}{\partial y} \cdot \Delta y = i \cdot \frac{\partial v}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial x} \cdot (i \cdot \Delta y)$$

again using a Cauchy-Riemann equation. Adding Δu and $i \cdot \Delta v$:

$$\begin{aligned}\Delta u + i \cdot \Delta v &= \frac{\partial u}{\partial x} \cdot \Delta x + i \cdot \frac{\partial v}{\partial x} \cdot (i \cdot \Delta y) + i \cdot \frac{\partial v}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial x} \cdot (i \cdot \Delta y) \\ &= \left[\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x}\right] \cdot \left[\Delta x + i \cdot \Delta y\right] = \left[\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x}\right] \cdot \Delta z\end{aligned}$$

Thus

$$\frac{f(w + \Delta z) - f(w)}{\Delta z} = \frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x}$$

Letting $\Delta z \to 0$, the partial derivatives approach their values at w, because those partial derivatives are continuous. This proves that $f'(w)$ exists. □

6. Taylor Series: Analytic Functions

Let a_n be a complex sequence and let $z_0 \in \mathbb{C}$. The formula

$$\sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n = \lim_{m \to \infty} \sum_{n=0}^{m} a_n \cdot (z - z_0)^n$$

is the *Taylor series for a_n about z_0*.

[8]Recall from Proposition 1 that the partial derivatives of u and v are evaluated at points in the disk. As $z \to w$, those points will approach w.

Let V be an open subset of $\mathbb{C}$ and let $f : V \to \mathbb{C}$. We say that f is *analytic* (and we say that f is an *analytic function*) if for every $w \in V$, there is $r > 0$ such that $D(w;r) \subseteq V$ and such that there is a complex sequence a_n with

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z-w)^n \quad \text{for every} \quad z \in D(w;r)$$

In other words, $f(z)$ has a Taylor series formula about every point in its domain.

It can well be that the sequence a_n in the Taylor series depends on the point w in question.

One of the main theorems in our course is that the analytic functions are precisely the holomorphic functions. In this section we prove that analytic functions are holomorphic; this leads to a host of examples.

Write the complex sequence a_n in rectangular form: $a_n = c_n + i \cdot d_n$. Then the series for a_n converges if and only if the series for c_n and for d_n converge. And it will not surprise us to find that

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c_n + i \cdot \sum_{n=0}^{\infty} d_n$$

Our first result is very basic: boundedness of absolute sums implies absolute convergence implies convergence.

Proposition 15. *Let $a_n \in \mathbb{C}$. Define*

$$b_n = \sum_{k=0}^{n} |a_k|$$

If the sequence b_n is bounded, then the series for $|a_n|$ converges, and the series for a_n converges.

Proof. We are assuming that the (real-valued) increasing sequence b_n is bounded; it converges by the Monotone Convergence Theorem. In other words, the series for $|a_n|$ converges.

Write $a_k = c_k + i \cdot d_k$, where $c_k, d_k \in \mathbb{R}$. Then $|c_k| \leq |a_k|$ for all k, and so the series for $|c_k|$ is bounded, and therefore converges. We have $0 \leq |c_k| - c_k \leq 2 \cdot |c_k|$, and this shows that the series for $|c_k| - c_k$ is increasing and bounded, and so that series converges. It follows that the series for $c_k = |c_k| - (|c_k| - c_k)$ converges.

Similarly, the series for d_k converges, and it follows that the series for a_k converges. $\square$

Now we make a definition that is key to telling whether a complex sequence a_n can be used in a Taylor series. Let r be a positive number; we say that r is a *radius* for a_n if, whenever $0 \leq s < r$, the sequence $|a_n| \cdot s^n$ is bounded.

Two trivial examples. if a_n is a non-zero constant and if $0 < r \leq 1$, then r is a radius for a_n. If $a_n = t^n$ for some non-zero complex number t, and if $0 < r \leq 1/|t|$, then r is a radius for a_n.

Here is the significance of the radius.

Proposition 16. *Let a_n be a complex sequence, and let $r > 0$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot z^n \tag{6}$$

converges for all $z \in D(0;r)$ if and only if r is a radius for a_n.

Proof. Assume that $f(z)$ converges for all $z \in D(0;r)$, and choose s with $0 \le s < r$. Then $s \in D(0;r)$, and

$$f(s) = \sum_{n=0}^{\infty} a_n \cdot s^n$$

We are assuming that this series converges, and so $a_n \cdot s^n \to 0$ as $n \to \infty$. It follows that $|a_n| \cdot s^n \to 0$, and therefore the sequence $|a_n| \cdot s^n$ is bounded. Thus, r is a radius.

Assume that r is a radius for a_n. Let $z \in D(0;r)$. There is s with $|z| < s < r$. Choose an upper bound B for the sequence $|a_n| \cdot s^n$. Let m be a positive integer and estimate

$$\begin{aligned}\sum_{n=0}^{m} |a_n \cdot z^n| &= \sum_{n=0}^{m} |a_n| \cdot |z|^n = \sum_{n=0}^{m} |a_n| \cdot s^n \cdot \frac{|z|^n}{s^n} \\ &= \sum_{n=0}^{m} |a_n| \cdot s^n \cdot \left(\frac{|z|}{s}\right)^n \le \sum_{n=0}^{m} B \cdot \left(\frac{|z|}{s}\right)^n\end{aligned}$$

This last series is seen to be geometric, and since $|z|/s < 1$, it converges.

$$\sum_{n=0}^{m} |a_n \cdot z^n| \le \frac{B}{1 - |z-w|/s}$$

By Proposition 15, the series $f(z)$ converges. □

You should remember the Ratio Test from calculus; it can often be used with Proposition 16.

Ratio Test. *Suppose that a_n is eventually not 0 (not 0 for n large enough), and suppose that*

$$\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = R$$

whether R is a number or $R = \infty$. Then R is a radius for a_n.

Proof. Taking $s < R$, there is a positive integer m such that if $n \ge m$, then $|a_n/a_{n+1}| > s$. It follows that

$$|a_m| \cdot s^m \ge |a_{m+n}| \cdot s^{m+n} \quad \text{for all } n \ge 0$$

This proves that $|a_n| \cdot s^n$ is bounded. □

If a_n has a ratio limit R as in the Ratio Test, then Proposition 16 says that the Taylor series (6) converges on $D(0;R)$.

Proposition 17. *Let $a_n \in \mathbb{C}$. Let r be a radius of a_n. Then r is a radius of the sequence $n \cdot a_n$.*

Proof. Let $0 \leq s < r$. Choose t with $s < t < r$. The sequence $|a_n| \cdot t^n$ is bounded above – say by B. Since $s/t < 1$, the sequence $n \cdot (s/t)^n$ goes to 0 as $n \to \infty$, as can be seen using the real exponential function.[9]

Then

$$|n \cdot a_n| \cdot s^n = |a_n| \cdot t^n \cdot \left(n \cdot \frac{s}{t}\right)^n \leq B \cdot \left(n \cdot \frac{s}{t}\right)^n$$

The sequence on the right goes to 0 as $n \to \infty$, and it follows that $|n \cdot a_n| \cdot s^n$ is bounded. □

We will need the following in a derivative calculation momentarily.

Proposition 18. *Let $r > 0$, and let $z, w \in D(0; r)$ with $z \neq w$, and let n be a positive integer. Then*

$$\left| \frac{z^n - w^n}{z - w} - n \cdot w^{n-1} \right| \leq |z - w| \cdot \frac{(n-1) \cdot n}{2} \cdot r^{n-2}$$

Proof. Direct calculation. When $n = 1$, the conclusion is trivial; let $n \geq 2$ and write S for the sum inside the norm bars.

$$\begin{aligned} S &= \frac{z^n - w^n}{z - w} - n \cdot w^{n-1} = \sum_{k=0}^{n-1} z^k \cdot w^{n-1-k} - n \cdot w^{n-1} \\ &= \sum_{k=0}^{n-1} \left[z^k \cdot w^{n-1-k} - w^{n-1} \right] = \sum_{k=0}^{n-1} \left[(z^k - w^k) \cdot w^{n-1-k} \right] \\ &= \sum_{k=1}^{n-1} \left[(z^k - w^k) \cdot w^{n-1-k} \right] = \sum_{k=1}^{n-1} \left[w^{n-1-k} \cdot (z - w) \cdot \sum_{j=0}^{k-1} z^j \cdot w^{k-1-j} \right] \\ &= (z - w) \cdot \sum_{k=1}^{n-1} \left[\sum_{j=0}^{k-1} z^j \cdot w^{n-2-j} \right] \end{aligned}$$

Now we switch the summations and let $p = k - 1$. We see that $0 \leq j \leq n - 2$; for each j, we have $j < k \leq n - 1$, so that $j \leq p \leq n - 2$. We obtain

$$S = (z - w) \cdot \sum_{j=0}^{n-2} \sum_{p=j}^{n-2} z^j \cdot w^{n-2-j} = (z - w) \cdot \sum_{j=0}^{n-2} (n - 1 - j) \cdot z^j \cdot w^{n-2-j}$$

Now we can estimate.

$$\begin{aligned} |S| &\leq |z - w| \cdot \left| \sum_{j=0}^{n-2} (n - 1 - j) \cdot z^j \cdot w^{n-2-j} \right| \\ &\leq |z - w| \cdot \sum_{j=0}^{n-2} (n - 1 - j) \cdot r^j \cdot r^{n-2-j} = |z - w| \cdot \frac{(n-1) \cdot n}{2} \cdot r^{n-2} \end{aligned}$$

□

[9]Write $n \cdot (s/t)^n$ as $\exp(\ln(n) + n \cdot \ln(s/t))$.

Now we are ready for the main theorem of this section. The formula for a_n in terms of the derivatives of $f(z)$ is called *Taylor's Formula.*

Theorem 19. *Let a_n be a complex sequence. Let r be a radius of a_n. Let $z_0 \in \mathbb{C}$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n$$

is holomorphic on $D(z_0; r)$. Indeed, for all $z \in D(z_0; r)$ we have

$$f'(z) = \sum_{n=1}^{\infty} a_n \cdot n \cdot (z - z_0)^{n-1}$$

In fact, $f(z)$ has infinitely many derivatives on $D(z_0; r)$, and we have

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \text{for all} \quad n$$

Proof. We first prove the theorem when $z_0 = 0$. Proposition 16 shows that $f(z)$ converges for all $z \in D(0; r)$. Define $g(z)$ to be the series on the right of the claimed formula for $f'(z)$. Proposition 17 shows that $g(z)$ converges on $D(0; r)$.

Let $w \in D(0; r)$. Choose s with $|w| < s < r$. For $z \in D(0; s) \setminus \{w\}$, The series for $f(z)$ and $f(w)$ can be combined.

$$\frac{f(z) - f(w)}{z - w} = \sum_{n=0}^{\infty} a_n \cdot \frac{z^n - w^n}{z - w}$$

The $n = 0$ and $n = 1$ terms are 0, and so we take n to start at 2. We can then write

$$\frac{f(z) - f(w)}{z - w} - g(w) = \sum_{n=2}^{\infty} a_n \cdot \left[\frac{z^n - w^n}{z - w} - n \cdot w^{n-1}\right]$$

We will show that this difference converges to 0 as $z \to w$.

We estimate, using Proposition 18, noting that $w, z \in D(0; s)$.

$$\begin{aligned}
\left|\sum_{n=2}^{\infty} a_n \cdot \left[\frac{z^n - w^n}{z - w} - n \cdot w^{n-1}\right]\right| &\le \sum_{n=2}^{\infty} |a_n| \cdot \left|\frac{z^n - w^n}{z - w} - n \cdot w^{n-1}\right| \\
&\le \sum_{n=2}^{\infty} |a_n| \cdot |z - w| \cdot \frac{(n-1) \cdot n}{2} \cdot s^{n-2} \\
&= |z - w| \cdot \sum_{n=2}^{\infty} |a_n| \cdot \frac{(n-1) \cdot n}{2} \cdot s^{n-2}
\end{aligned}$$

Since r is a radius of a_n, Proposition 16 shows that the series for $|a_n| \cdot s^n$ converges. Proposition 17 shows that r is a radius for $(n-1) \cdot n \cdot |a_n|/2$, and so the series

$$\frac{n \cdot (n-1)}{2} \cdot |a_n| \cdot s^{n-2}$$

converges as well – call the sum T. We have shown that

$$\left|\frac{f(z) - f(w)}{z - w} - g(w)\right| \le |z - w| \cdot T$$

As $z \to w$, the right side goes to 0, and thus

$$\lim_{z\to w} \frac{f(z) - f(w)}{z - w} = g(w)$$

and this proves the theorem in case $z_0 = 0$.

For the general case, let

$$g(z) = \sum_{n=0}^{\infty} a_n \cdot z^n$$

so that $g(z)$ is holomorphic on $D(0; r)$ and that $g'(z)$ can be computed term by term. We see that $f(z) = g(z - z_0)$ is then holomorphic on $D(z_0; r)$.

Now we can apply to $f'(z)$ what we did to $f(z)$: we get the existence of $f''(z)$, and so on. We see that $f(z)$ has infinitely many derivatives on $D(z_0; r)$. The formula for the coefficients follows easily from the fact that $f(z_0) = a_0$. □

7. Exponential, Cosine, Sine, Logarithm

The most important function in mathematics: define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

We see that $\exp(0) = 1$. The Ratio Test shows that every positive number is a radius of $1/n!$, and so the series converges on all of $\mathbb{C}$. In other words, $\exp(z)$ is an *entire function*. Since we can differentiate term by term, we find that $\exp'(z) = \exp(z)$.

Let $a \in \mathbb{C}$ and compute that $\exp(z + a) \cdot \exp(-z)$ has derivative 0 for all z. Proposition 13 shows that $\exp(z + a) \cdot \exp(-z)$ is a constant. Taking $z = 0$, we see that

$$\exp(z + a) \cdot \exp(-z) = \exp(a) \tag{7}$$

Taking $a = 0$ in (7), we have $\exp(z) \cdot \exp(-z) = 1$. It follows that $\exp(z)$ is never 0; and we have $\exp(-z) = 1/\exp(z)$.

In (7), write $c = z + a$ and $d = -z$, so that $a = c - z = c + d$ and we have the expected identity

$$\exp(c) \cdot \exp(d) = \exp(c + d)$$

When $z \in \mathbb{R}$, the formula for $\exp(z)$ gives the usual real exponential function. By analogy with the real case, we continue to write $\exp(z) = e^z$ for $z \in \mathbb{C}$. This formula makes algebraic sense when z is a (real) rational number.

We construct the trigonometric functions in a similar manner, using the coefficients of their real series. Define

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!} \quad \text{and} \quad \sin(z) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!}$$

As with e^z, every positive number is a radius, and so $\cos(z)$ and $\sin(z)$ are entire functions. We get the expected formulas

$$\cos'(z) = -\sin(z) \quad \text{and} \quad \sin'(z) = \cos(z)$$

We also see that $\cos(0) = 1$ and $\sin(0) = 0$.

It follows that $\cos^2(z)+\sin^2(z)$ has derivative 0 for all z; Proposition 13 says that this function is constant. The case $z=0$ shows us that the constant is 1, and we have the Pythagorean Identity.

$$\cos^2(z)+\sin^2(z)=1$$

Next, another central fact. It is most common to use the following in the case that $z\in\mathbb{R}$, but it holds in general.

Euler's Identity. *For all $z\in\mathbb{C}$ we have*

$$e^{i\cdot z}=\cos(z)+i\cdot\sin(z)\quad\text{for all}\quad z\in\mathbb{C}$$

Proof. We have to remember how to compute powers of i.

$$\begin{aligned}\exp(i\cdot z)&=\sum_{n=0}^{\infty}\frac{i^n\cdot z^n}{n!}=\sum_{n=0}^{\infty}\frac{i^{2n}\cdot z^{2n}}{(2n)!}+\sum_{n=0}^{\infty}\frac{i^{2n+1}\cdot z^{2n+1}}{(2n+1)!}\\&=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot z^{2n}}{(2n)!}+i\cdot\sum_{n=0}^{\infty}\frac{(-1)^n\cdot z^{2n+1}}{(2n+1)!}=\cos(z)+i\cdot\sin(z)\end{aligned}$$

□

Earlier in the course, we defined

$$\exp(i\cdot\theta)=\cos(\theta)+i\cdot\sin(\theta)$$

Now that we have the complex exponential function, we interpret that equation as a property of the exponential.

Here is a consequence: let $x,y\in\mathbb{R}$ and compute

$$\exp(x+i\cdot y)=\exp(x)\cdot\exp(i\cdot y)=\exp(x)\cdot(\cos(y)+i\cdot\sin(y))$$

This exhibits the real and imaginary parts of $\exp(x+iy)$.

Here is a collection of identities that follow.

(1) $\overline{\exp(z)}=\exp(\overline{z})$ (complex conjugate)
(2) If $x,y\in\mathbb{R}$, then $|\exp(x+i\cdot y)|=\exp(x)$
(3) Let $w,z\in\mathbb{C}$. Then $\exp(w)=\exp(z)$ if and only if $w-z=2\pi ki$ for some integer k.

Another consequence: we can write $\cos(z)$ and $\sin(z)$ in terms of $\exp(z)$. It is a direct calculation to prove these equations:

$$\cos(z)=\frac{\exp(i\cdot z)+\exp(-i\cdot z)}{2}\quad\text{and}\quad\sin(z)=\frac{\exp(i\cdot z)-\exp(-i\cdot z)}{2\cdot i}$$

The cosine and sine angle addition formulas follow directly. Let $a,b\in\mathbb{C}$.

$$\begin{aligned}\cos(a+b)&=\cos(a)\cdot\cos(b)-\sin(a)\cdot\sin(b)\\\sin(a+b)&=\sin(a)\cdot\cos(b)+\cos(a)\cdot\sin(b)\end{aligned}$$

The point is that we have the familiar identities from the real case even when the variables are complex.

A Logarithm

We are interested in the logarithm; this turns out to be ticklish. We will give one example and discuss others in class. We will use some particular open subsets of the complex numbers. Let S_1 consist of the complex numbers with positive real part. Let S_2 be the set of complex numbers with positive imaginary part. Let S_3 consist of complex numbers with negative imaginary part. Finally, let $S = S_1 \cup S_2 \cup S_3$. Then S can be described as the complex plane with the negative x-axis and 0 removed.

In a previous problem, you considered the function

$$A(x + i \cdot y) = \arctan(y/x)$$

on S_1, and you showed that

$$L(z) = \ln|z| + i \cdot A(z)$$

satisfies the Cauchy-Riemann equations and that the first partial derivatives are continuous. Thus, $L(z)$ defines a holomorphic function on S_1. We also see that $A(z) \in \operatorname{Arg}(z)$ for all $z \in S_1$.

For $z \in S_1$ we compute

$$\exp(L(z)) = \exp(\ln|z|) \cdot \exp(i \cdot A(z)) = |z| \cdot \exp(i \cdot A(z)) = z$$

For $z \in T_1$ (from the problem just given), we see that $\exp(z) \in S_1$, for the real part of $\exp(z)$ is $|z| \cdot \cos(y)$, where $-\pi/2 < y < \pi/2$ shows that $\cos(y) > 0$. Thus, $L(\exp(z))$ is defined for all $z \in T_1$. Write such $z = x + i \cdot y$ in rectangular, and then

$$L(\exp(z)) = \ln|\exp(z)| + i \cdot A(\exp(z)) \ln|e^x| + i \cdot y = x + i \cdot y = z$$

We have shown that $L(z)$ and $\exp(z)$ are inverse functions here. We will write $\log(z) = L(z)$.

We an perform a similar construction on the set S_2. This time, the argument function needs to use the inverse-cotangent.[10] For $z \in S_2$, write $z = x + i \cdot y$ and define

$$A(z) = \cot^{-1}(x/y)$$

Again, we have a logarithm that inverts the exponential.

$$\log(z) = \ln|z| + i \cdot A(z) \quad \text{for all} \quad z \in S_2$$

This function log maps S_2 onto T_2, the set of $x + i \cdot y$ with $0 < y < \pi$.

Taking $S_1 \cup S_2$, we now have a logarithm on a fairly large open set. We can extend to the set S_3, by defining

$$A(x + i \cdot y) = \cot^{-1}(x/y) - \pi$$

You can observe that A maps S_3 into the real open interval $(-\pi, 0)$. The resulting logarithm maps S_3 onto the set T_3 of complex numbers $x + i \cdot y$ with $-\pi < y < 0$. This logarithm agrees with the one on S_1 where they overlap.

Here is a summary. Define S to be the complex numbers that are not non-positive numbers. In other words, $x + i \cdot y \in S$ if and only if either $y \neq 0$ or $x > 0$. (The set

[10]The *inverse cotangent* function $\cot^{-1}(t)$ gives the angle α such that $0 < \alpha < \pi$ and $\cot(\alpha) = t$. It is defined for all real numbers t. Notice that $\sin(\alpha) > 0$ here.

S is $\mathbb{C}$ with 0 and the negative x-axis removed.) Define $A : S \to (-\pi, \pi)$, as follows:

$$A(x + i \cdot y) = \begin{cases} \arctan(y/x) & \text{if } x > 0 \\ \cot^{-1}(x/y) & \text{if } y > 0 \\ \cot^{-1}(x/y) - \pi & \text{if } y < 0 \end{cases}$$

Then

$$\log(z) = \ln|z| + i \cdot A(z)$$

defines a holomorphic mapping from S onto the set T of complex numbers $x + i \cdot y$ with $-\pi < y < y$. Then exp maps T onto S and

$$\exp(\log(z)) = z \quad \text{for all} \quad z \in S \quad \text{and} \quad \log(\exp(z)) = z \quad \text{for all} \quad z \in T$$

What about Taylor series for $\log(z)$? First we work in $D(1; 1) \subset S$.

Consider the sequence $(-1)^n/n$. By the Ratio Test, 1 is a radius for this sequence, and so

$$L(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot (z-1)^n$$

is holomorphic on $D(1; 1)$. Notice that $L(1) = 0$.

Compute for $z \in D(1; 1)$ that

$$L'(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot (z-1)^{n-1} = \sum_{n=1}^{\infty} (1-z)^{n-1}$$

This geometric series converges to

$$\frac{1}{1-(1-z)} = \frac{1}{z}$$

Therefore, $(\log(z) - L(z))' = 0$ for all $z \in D(1; 1)$. Proposition 13 shows that $\log(z) - L(z)$ is constant, and since $\log(1) = L(1) = 0$, we have $\log(z) = L(z)$. Thus,

$$\log(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot (z-1)^n \quad \text{for all} \quad z \in D(1; 1)$$

Proposition 20. *The function* $\log(z)$ *is analytic on* S.

Proof. Let $w \in S$ and get $r > 0$ (as in a problem above) with $D(w; r) \subset S$. For $z \in D(w; r)$, a problem has shown that $z/w \in D(1; 1)$, and so the Taylor series for the logarithm on $D(1; 1)$ shows that

$$\log(z/w) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \left(\frac{z}{w} - 1\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \frac{1}{w^n} \cdot (z-w)^n$$

Another problem showed that $\log(z/w) = \log(z) - \log(w)$ for all $z \in D(w; r)$, and now we see that

$$\log(z) = \log(w) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \frac{1}{w^n} \cdot (z-w)^n \quad \text{for all} \quad z \in D(w; r) \tag{8}$$

□

A Square Root

We can use our logarithm to get a square root. Recall the sets

$$S = \{x + i \cdot y \mid x, y \in \mathbb{R},\ (x > 0 \text{ or } y \neq 0)\}$$

$$T_1 = \{x + i \cdot y \mid x, y \in \mathbb{R},\ -\pi/2 < y < \pi/2\}$$

The function $\log(z)/2$ maps S onto T_1. Define

$$R(z) = \exp(\log(z)/2) \quad \text{for} \quad z \in S$$

and R maps S to S_1. We see that

$$\big[R(z)\big]^2 = \exp(\log(z)) = z$$

so that $R(z)$ is a holomorphic square root.

Arbitrary Powers

You might remember that for real numbers a, b, with $b > 0$, we have

$$b^a = \exp(a \cdot \ln(b))$$

We want to explore an analogous complex formula.

We will continue to use the set S of complex numbers with the non-positive x-axis removed. For each $\alpha \in \mathbb{C}$, we want to define

$$z^\alpha = \exp(\alpha \cdot \log(z)) \quad \text{for all} \quad z \in S$$

The algebraic properties of the exponential function tell us for $\alpha, \beta \in \mathbb{C}$ that

$$z^\alpha \cdot z^\beta = z^{\alpha+\beta} \tag{9}$$

We also have $z^0 = 1$ and $1^\alpha = 1$.

Taking the derivative, we have

$$(z^\alpha)' = \exp(\alpha \cdot \log(z)) \cdot \frac{\alpha}{z} = \alpha \cdot z^{\alpha - 1}$$

The Power Rule! And notice that it holds for all complex number exponents – of course that includes all the cases where the exponent has algebraic significance, such as when α is an integer or rational number. We have constructed a holomorphic z^α on S.

Let's show that z^α is analytic on S. We will get a series on the disk $D(1;1)$ and then show how to transfer the series to other disks, exactly as we did with the logarithm. Our series needs the *binomial sequence*: define

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{n+1} = \frac{\alpha - n}{n+1} \cdot \binom{\alpha}{n} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

(Thus, there is a binomial sequence for each given complex number α.) Now define

$$L(z) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \cdot (z-1)^n$$

In the case that α is not a non-negative integer, a problem showed that the ratio limit of the binomial coefficients is 1. Theorem 19 proves that $L(z)$ can be differentiated term by term on $D(1;1)$. For $z \in D(1;1)$, it will be convenient to calculate $z \cdot L'(z)$. Here goes.

$$\begin{aligned}
z \cdot L'(z) = z \cdot \sum_{n=1}^{\infty} \binom{\alpha}{n} \cdot n \cdot (z-1)^{n-1} \\
&= (z-1) \cdot \sum_{n=1}^{\infty} \binom{\alpha}{n} \cdot n \cdot (z-1)^{n-1} + \sum_{n=1}^{\infty} \binom{\alpha}{n} \cdot n \cdot (z-1)^{n-1} \\
&= \sum_{n=1}^{\infty} \binom{\alpha}{n} \cdot n \cdot (z-1)^{n} + \sum_{n=1}^{\infty} \binom{\alpha}{n} \cdot n \cdot (z-1)^{n-1} \\
&= \sum_{n=1}^{\infty} \binom{\alpha}{n} \cdot n \cdot (z-1)^{n} + \sum_{n=0}^{\infty} \binom{\alpha}{n+1} \cdot (n+1) \cdot (z-1)^{n} \\
&= \sum_{n=0}^{\infty} \binom{\alpha}{n} \cdot n \cdot (z-1)^{n} + \sum_{n=0}^{\infty} \binom{\alpha}{n} \cdot (\alpha - n) \cdot (z-1)^{n} \\
&= \alpha \cdot L(z)
\end{aligned}$$

Now we can show that $z^\alpha = L(z)$ on $D(1;1)$. Consider

$$\begin{aligned}
z \cdot [z^{-\alpha} \cdot L(z)]' &= z \cdot \left[-\alpha \cdot z^{-\alpha-1} \cdot L(z) + z^{\alpha} \cdot L'(z)\right] \\
&= -\alpha \cdot z^{-\alpha} \cdot L(z) + z^{\alpha} \cdot \alpha \cdot L(z) \\
&= 0
\end{aligned}$$

Since $z \in D(1;1)$ implies that $z \neq 0$, we see that $z^{-\alpha} \cdot L(z)$ has 0 derivative, and Proposition 13 says it is constant. Taking $z = 1$, we have $L(1) = 1$, and thus, $z^{-\alpha} \cdot L(z) = 1$, so that

$$z^\alpha = L(z) \quad \text{on} \quad D(1;1)$$

We will drop the notation $L(z)$ and use z^α, remembering that the series is defined on $D(1;1)$.

Let $w \in S$, and we will see that it is easy to get a series for z^α in a disk around w. Indeed, recall the open set $S(w)$ of $z \in S$ such that $z/w \in S$.

Using the previous problem we are in position to show that z^α is analytic on S. Given $w \in S$ and $D(w;r) \subset S$, as previously, we have

$$z^\alpha = w^\alpha \cdot \sum_{n=0}^{\infty} \binom{\alpha}{n} \cdot \frac{1}{w^n} \cdot (z-w)^n \quad \text{for all} \quad z \in D(w;r) \tag{10}$$

8. Line Integrals

We begin working toward another of our fundamental theorems; Cauchy's Theorem involves integration of a holomorphic function over a curve in the complex plane. To handle curves generally, we would need some non-trivial topology that we would rather not broach. To avoid the topology we will restrict ourselves somewhat while still obtaining fairly strong versions of our main theorems.

The reader should be familiar with smooth curves in the plane. Since the plane is the set of complex numbers, a *smooth curve* is a function $g : [a,b] \to \mathbb{C}$, where $[a,b]$ is a closed interval in the reals, such that g' is continuous. If we write $g(t) = x(t)+i\cdot y(t)$ in rectangular form, then the definition we gave of the general derivative shows that

$$g'(t) = x'(t) + i \cdot y'(t)$$

for all $t \in [a, b]$. The *image* of a smooth curve is its function image: the set of all $g(t) \in \mathbb{C}$ such that $t \in [a, b]$. To call attention to the endpoints, we say that g *starts at* $g(a)$ *and ends at* $g(b)$.

It is very common to identify the curve g with its image. For instance, if $g(t) = t$ for $0 \le t \le 1$, then the image is a line segment on the x-axis. It is sometimes safe to confuse the function with its image, but to be formal, the *curve* is the function g, not its image.

A standard curve for a line segment: given $p, q \in \mathbb{C}$, define $L(p, q) : [0, 1] \to \mathbb{C}$ by the formula

$$L(p, q)(t) = p + t \cdot (q - p) \quad \text{for} \quad t \in [0, 1]$$

The image of L is the line segment connecting p and q.

To travel around a circle, choose a center $p \in \mathbb{C}$, a radius $r > 0$. The image of the smooth curve $p + r \cdot e^{it}$, for $0 \le t \le 2\pi$, is the circle with center p and radius r. We call this curve $C(p; r)$.

We occasionally want to travel over part of a circle: for $c \in \mathbb{C}$ and $\alpha < \beta$ and $r > 0$, define $\operatorname{arc}(c, r, \alpha, \beta)$ to be the curve mapping t to $c+r\cdot\exp(i\cdot t)$, for $\alpha \le t \le \beta$. It will be convenient to allow $\alpha > \beta$, as well; in that case, define $\operatorname{arc}(c, r, \alpha, \beta) = -\operatorname{arc}(c, r, \beta, \alpha)$. The image of the arc connects the two angles in every case.

It will be convenient to be able to "run g backwards." To this effect, we define the function $-g : [a, b] \to \mathbb{C}$ by the formula $-g(t) = g(a + b - t)$. Observe that if $a \le t \le b$, then $a \le a + b - t \le b$, as well, and so the expression $g(a + b - t)$ makes sense – in particular notice that g and $-g$ have the same image. Also notice that $-g$ is not $(-1) \cdot g$. This misuse of the negative sign is standard when working with curves.

Compute $(-g(t))' = (-1) \cdot g'(a + b - t)$; this shows that $-g$ is smooth.

As in calculus, we can define the *length* of a smooth curve:

$$|g| = \int_a^b |g'(t)| \cdot dt$$

Since g' is continuous, $|g'|$ is continuous, hence the integral is defined.

We want to integrate a continuous, complex function over a smooth curve. To get started, we suppose that $h : [a, b] \to \mathbb{C}$ is continuous. If x and y are the real and imaginary parts of h, respectively, then x and y are continuous, real-valued functions on $[a, b]$, and so their integrals are defined as in calculus:

$$\int_a^b x(t) \cdot dt \qquad \text{and} \qquad \int_a^b y(t) \cdot dt$$

We define

$$\int_a^b h(t) \cdot dt = \int_a^b x(t) \cdot dt + i \cdot \int_a^b y(t) \cdot dt$$

There are antiderivatives $X(t)$ for $x(t)$ and $Y(t)$ for $y(t)$, and we see that

$$\int_a^b h(t) \cdot dt = (X(b) - X(a)) + i \cdot (Y(b) - Y(a)) \tag{11}$$

If we let $H(t) = X(t) + i \cdot Y(t)$, then our formula for general derivatives shows that $H'(t) = h(t)$. The equation (11) can be written like this:

$$\int_a^b h(t) \cdot dt = H(b) - H(a) \tag{12}$$

and we have a natural generalization of the Fundamental Theorem of Calculus. To summarize: we have defined the integral of a continuous function $h : [a, b] \to \mathbb{C}$. If $H : [a, b] \to \mathbb{C}$ with $H'(t) = h(t)$, then we can integrate h using H as in real calculus.

Notice that none of the functions we have considered up to this point are *holomorphic*, since we have been taking a closed interval in the reals as domain.

Let $g : [a, b] \to \mathbb{C}$ be a smooth curve, and suppose that f is continuous on the image of g. We consider the function $f(g(t)) \cdot g'(t)$, which maps $[a, b]$ into $\mathbb{C}$; it is continuous, because f and g and g' are continuous, and therefore, its integral is defined. We write

$$\int_g f(z) \cdot dz = \int_a^b f(g(t)) \cdot g'(t) \cdot dt \tag{13}$$

This is called the *line integral of f over g*. In this notation, $f(z) \cdot dz$ is a shorthand: $z = g(t)$ and $dz = g'(t) \cdot dt$ in the standard differential notation.

Line integrals are sometimes called *contour* integrals, the word contour referring to the image of a smooth curve.

If we write $g(t) = x(t) + i \cdot y(t)$ and $f(z) = u(z) + i \cdot v(z)$ to indicate real and imaginary parts of each function, then we can write the line integral (13) in terms of ordinary real integrals:

$$\begin{aligned} \int_g f(z) \cdot dz = &\int_a^b [u(g(t)) \cdot x'(t) - v(g(t)) \cdot y'(t)] \cdot dt \\ &+ i \cdot \int_a^b [u(g(t)) \cdot y'(t) + v(g(t)) \cdot x'(t)] \cdot dt \end{aligned}$$

This looks cumbersome, but it points the way to evaluation of line integrals: do orginary algebra on the integrand to reduce the calculation to real-valued integrals.

We have been careful about definitions. As in multi-variable calculus, the purpose of the line integral notation is to hide cumbersome definitions behind more ordinary algebra – we will see examples momentarily.

Here are properties of the line integral meant to show the naturalness of its notation by imitating properties of ordinary integrals on the real line. Until further notice, we assume that $g : [a, b] \to \mathbb{C}$ is a smooth curve. The proofs of the first two properties are left to you.

Property 1. Let $f(z)$, $k(z)$ be continuous, complex-valued functions on the image of g. Then

$$\int_g (f(z) + k(z)) \cdot dz = \int_g f(z) \cdot dz + \int_g k(z) \cdot dz$$

Property 2. Let $f(z)$ be a continuous, complex-valued function on the image of g, and let $\alpha \in \mathbb{C}$. Then

$$\int_g (\alpha \cdot f(z)) \cdot dz = \alpha \cdot \int_g f(z) \cdot dz$$

(Note: α and $f(g(t)) \cdot g'(t)$ have real and imaginary parts – you will need to use them to apply the definition.)

Property 3. Let $f(z)$ be a continuous, complex-valued function on the image of g. Then

$$\int_{-g} f = -\int_g f$$

Proof. Indeed, observe that

$$f(-g(t)) \cdot (-g)'(t) = f(g(a+b-t)) \cdot (-1)g'(a+b-t)$$

Let $s = a+b-t$, and then

$$f(-g(t)) \cdot (-g)'(t) = (-1) \cdot f(g(s)) \cdot g'(s)$$

Then we use Property 2 to factor out -1.

$$\int_{-g} f(z) \cdot dz = \int_a^b f(-g(t)) \cdot g'(t) \cdot dt = \int_b^a (-1) f(g(s)) \cdot g'(s) \cdot (-ds)$$
$$= (-1) \int_a^b f(g(s)) \cdot g'(s) \cdot ds = -\int_g f(z) \cdot dz$$

□

Property 4. Let $f(z)$ be a continuous, complex-valued function on the image of g, and suppose that M is a real number with $|f(g(t))| \leq M$ for all $t \in [a,b]$. Then

$$\left| \int_g f(z) \cdot dz \right| \leq M \cdot |g|$$

Proof. Let θ be an argument for the complex number $\int_g f(z) \cdot dz$, and then

$$\left| \int_g f(z) \cdot dz \right| = e^{-i\theta} \cdot \int_g f(z) \cdot dz$$

In particular, the number on the right is a real number, and therefore is equal to its real part. By Property 2 we can pull the $e^{-i\theta}$ into the integral.

$$\left| \int_g f(z) \cdot dz \right| = \mathrm{Re}\left[e^{-i\theta} \int_g f(z) \cdot dz \right] = \mathrm{Re}\left[\int_g e^{-i\theta} \cdot f(z) \cdot dz \right]$$

The definition of integration over a curve in the complex plane shows that the real part of the integral is the integral of the real part of the integrand:

$$\mathrm{Re}\left[\int_g e^{-i\theta} \cdot f(z) \cdot dz \right] = \int_a^b \mathrm{Re}[e^{-i\theta} \cdot f(g(t)) \cdot g'(t)] \cdot dt \tag{14}$$

We know that the real part of a complex number is less than or equal to its modulus. Thus,

$$\mathrm{Re}[e^{-i\theta} \cdot f(g(t)) \cdot g'(t)] \leq \left| e^{-i\theta} \cdot f(g(t)) \cdot g'(t) \right|$$

The integral on the right side of (14) is an ordinary real integral. Thus, we can use the inequality just established to show that

$$\int_a^b \mathrm{Re}[e^{-i\theta} \cdot f(g(t)) \cdot g'(t)] \cdot dt \leq \int_a^b \left|e^{-i\theta} \cdot f(g(t)) \cdot g'(t)\right| \cdot dt$$

Continuing this estimate, we use that $|e^{-i\theta}| = 1$, and that $|f(g(t))| \leq M$.

$$\int_a^b \left|e^{-i\theta} \cdot f(g(t)) \cdot g'(t)\right| \cdot dt \leq \int_a^b M \cdot |g'(t)| \cdot dt = M \cdot |g|$$

Putting all this together, we have the desired inequality. □

Our final fact about line integration over the smooth curve g looks like the Fundamental Theorem of Calculus. It is a special case of (12).

Property 5. Let $f(z)$ be a continuous, complex-valued function on the image of g, and suppose that it has an antiderivative $F(z)$ there.[11] Then

$$\int_g f(z) \cdot dz = F(g(b)) - F(g(a))$$

Proof. The general Chain Rule shows us that

$$\big[F(g(t))\big]' = F'(g(t)) \cdot g'(t) = f(g(t)) \cdot g'(t)$$

and the equation follows from (12) on p.24. □

In the context of Property 5, if the smooth curve $g(t)$ starts and ends at the same point, then $g(a) = g(b)$, and so $F(g(b)) - F(g(b)) = 0$. This is very important, as we will see.

Property 5 should not be misinterpreted: it does not say that the line integral depends only on starting and ending points.

We can use integral substitution on line integrals. Here are the technical details. Let $g : [a, b] \to [c, d]$ and $h : [c, d] \to \mathbb{C}$, let g and h be smooth. Then

$$\int_h f(z) \cdot dz = \int_g f(h(w)) \cdot h'(w) \cdot dw$$

Indeed, the two integrals are these:

$$\int_c^d f(h(t)) \cdot h'(t) \cdot dt = \int_a^b f(h(g(s))) \cdot h'(g(s)) \cdot g'(s) \cdot ds$$

The equality of these last two integrals follows directly from (12) and the Chain Rule.

We want to generalize line integration from smooth curves to multi-sets of smooth curves. A *multi-set* is like a set, except that repeats are allowed (and counted). To distinguish multi-sets from sets, we will enclose them in angle brackets. Thus, the set $\{a, a\}$ is equal to the set $\{a\}$, but the multi-sets $< a, a >$ and $< a >$ are different. Like a set, there is no ordering of elements (constituents) in a multi-set, and so the multi-sets $< a, a, b >$ and $< a, b, a >$ are equal. In the multi-set $< g_1, g_2, \ldots, g_n >$, if

[11] Technical note: for $F'(z) = f(z)$ to be defined when z is in the image of g, we need z to be a limit point of the domain of F. If g is not constant, then every element of the image of g is a limit point of the image, and so we only need F to be defined on the image of g. We will neither prove nor use this fact.

we replace one of the g_k by a different object, we have changed the multi-set. If we re-order the g_k, we have not changed the multi-set.

A *chain* is a non-empty multi-set of smooth curves:

$$G =< g_1, g_2, \dots, g_n >$$

such that each curve ends where the next one starts. More formally, let the domain of each g_k be $[a_k, b_k]$, and we want to have $g_k(b_k) = g_{k+1}(a_{k+1})$ when $1 \leq k < n$. If, additionally, $g_1(a_1) = g_n(b_n)$, (if the chain starts and ends at the same point), then we call G a *closed chain*.

The *image* of the chain G is the union of the images of its constituents. The *length* of G is denoted $|G|$ and

$$|G| = \sum_{k=1}^{n} |g_k|$$

The circle $< C(c;r) >$ makes a closed chain with a single constituent. If $p, q, r \in \mathbb{C}$, the chain $< L(p,q), L(q,r), L(r,p) >$ is a closed chain; it is called a *chain triangle* for obvious reasons. (We allow p, q, r to be co-linear, by the way, and in that case the image of the chain triangle is a line segment.)

Let $G =< g_1, \dots, g_n >$ be a chain, and suppose that f is continuous on the image of G. Define

$$\int_G f(z) \cdot dz = \sum_{k=1}^{n} \int_{g_k} f(z) \cdot dz$$

This is the *line integral of f over G*.

We can easily generalize to chains the properties 1-5 we proved about integration over smooth curves.

Here is a more technical fact allowing us to switch a limit and an integral.

Proposition 21. *Let $A \subseteq \mathbb{C}$ be open and let G be a chain whose image lies in A. Let $f_n : V \to \mathbb{C}$ for $n = 0, 1, 2, \dots$ each be continuous, and suppose that $f_n \to f$ uniformly on A as $n \to \infty$. Then*

$$\lim_{n \to \infty} \int_G f_n(z) \cdot dz = \int_G f(z) \cdot dz$$

Proof. By Proposition 11, $f(z)$ is continuous on A, and so the integral of $f(z)$ over the chain G is defined. To get the limit of the conclusion, it suffices to get the analogous limit over a smooth curve g that is a constituent of the chain.

Let $\epsilon > 0$ and get N such that $n \geq N$ implies $|f_n(z) - f(z)| < \epsilon$ for all $x \in [a, b]$. For such n estimate

$$\begin{aligned} \left| \int_g f(z) \cdot dz - \int_g f_n(z) \cdot dz \right| &= \left| \int_g (f(z) - f_n(z)) \cdot dz \right| \\ &\leq \epsilon \cdot |g| \end{aligned}$$

Since ϵ is arbitrary, the proof is complete. □

9. Goursat's Theorem

This technical result is our entry into Cauchy's Theorem. It presents a level of sophistication above what we have done up to this point. Recall the definition of a chain triangle. The *closed interior* of a chain triangle is the boundary and interior of the image of that triangle in the complex plane.

Goursat's Theorem. *Let V be an open subset of $\mathbb{C}$ and let $f : V \to \mathbb{C}$ be holomorphic. Suppose that T is a chain triangle whose closed interior lies in V. Then $\int_T f(z) \cdot dz = 0$.*

Proof. Suppose that $T =< L(p,q), L(q,r), L(r,p) >$. Let R be the midpoint of the segment from p to q, let P be the midpoint of the segment qr, let Q be the midpoint of the segment rp. We will use these points to divide T into four sub-triangles. Define

$$\begin{aligned} A &=< L(R,Q), L(Q,p), L(p,R) > \\ B &=< L(Q,P), L(P,r), L(r,Q) > \\ C &=< L(P,R), L(R,q), L(q,P) > \\ D &=< L(R,P), L(P,Q), L(Q,R) > \end{aligned}$$

The closed interiors of these four chain triangles are contained in the closed interior of T. We claim that the line integral of $f(z)$ over T is the sum of the line integrals over these four sub-triangles. This is a direct calculation, using a previous problem in such manipulations as

$$\int_{L(p,R)} f(z) \cdot dz + \int_{L(R,q)} f(z) \cdot dz = \int_{L(p,q)} f(z) \cdot dz$$

and using another problem to conclude that reversing a segment negates the integral – for instance $L(R,Q) = -L(Q,R)$, so that Property 3 applies.

$$\begin{aligned} \int_{L(R,Q)} f(z) \cdot dz + \int_{L(Q,R)} f(z) \cdot dz &= \int_{L(R,Q)} f(z) \cdot dz + \int_{-L(R,Q)} f(z) \cdot dz \\ &= \int_{L(R,Q)} f(z) \cdot dz - \int_{L(R,Q)} f(z) \cdot dz = 0 \end{aligned}$$

When the 12 line integrals over line segments are added up, we end up with the line integral over T. Thus,

$$\int_T f(z) \cdot dz = \int_A f(z) \cdot dz + \int_B f(z) \cdot dz + \int_C f(z) \cdot dz + \int_D f(z) \cdot dz \tag{15}$$

Next we claim that one of the four sub-triangles, call it E, satisfies

$$\left| \int_E f(z) \cdot dz \right| \geq \frac{1}{4} \cdot \left| \int_T f(z) \cdot dz \right|$$

Indeed, if each sub-integral is less than one-fourth of the integral over T, then the sum of the four numbers on the right side of (15) cannot be the integral over T. We call the sub-triangle E a *fat sub-triangle*.

Now we define a sequence of triangles. Let $T_0 = T$; let T_1 be a fat sub-triangle of T_0; let T_2 be a fat sub-triangle of T_1; and so on. An easy induction argument shows that

$$\frac{1}{4^k} \cdot \left| \int_T f(z) \cdot dz \right| \leq \left| \int_{T_k} f(z) \cdot dz \right| \tag{16}$$

for each $k \geq 0$.

Plane geometry shows that each of the four sub-triangles of some T_k has length exactly one-half the length of T_k. In other words, $|T_{k+1}| = |T_k|/2$ for all $k \geq 0$. It follows that

$$|T_k| = \frac{1}{2^k} \cdot |T_0| \tag{17}$$

The closed interior of each T_k is a closed and bounded subset of the complex numbers, and the closed interior of T_k contains the closed interior of T_{k+1}. Proposition 7 finds a complex number c in all the T_k. Since the T_k's are all in T, and since $T \subset V$, we have $c \in V$, and so $f'(c)$ is defined.

Let ϵ be an arbitrary positive number. We will obtain the estimate

$$\left| \int_{T_0} f(z) \cdot dz \right| \leq \epsilon \cdot |T_0|^2 \tag{18}$$

Because ϵ is arbitrary, this will prove that the line integral is 0, as needed.

We will identify a particular T_k and then work back to T_0. Because $f'(c)$ is defined, there is $\delta > 0$ such that if $z \in D(c; \delta)$, then

$$|f(z) - f(c) - (z - c) \cdot f'(c)| \leq \epsilon \cdot |z - c|$$

The sides of the image triangle of T_{k+1} are half as big as the sides of T_k, for each k. It follows that the dimensions of T_k go to 0 as $k \to \infty$. Thus, there is a positive integer k such that the closed interior of T_k is contained in $D(c; \delta)$. When z is in the closed interior of T_k, the estimate just made holds. In the right side $\epsilon \cdot |z - c|$, we have $|z - c| \leq |T_k|$, and we see that

$$|f(z) - f(c) - (z - c) \cdot f'(c)| \leq \epsilon \cdot |T_k|$$

Property 4 then yields

$$\left| \int_{T_k} \Big(f(z) - f(c) - (z - c) \cdot f'(c) \Big) \cdot dz \right| \leq \epsilon \cdot |T_k| \cdot |T_k| \tag{19}$$

Next we rid the integral of $f(c) + (z - c) \cdot f'(c)$. Indeed, this function has an antiderivative $f(c) \cdot z + (z - c)^2 \cdot f'(c)/2$ on the entire complex plane. By Property 5, we can integrate this function over each line segment side of T_k by taking the difference in this antiderivative. Since T_k is a closed chain, this proves that

$$\int_{T_k} \Big(f(c) + (z - c) \cdot f'(c) \Big) \cdot dz = 0$$

Therefore,

$$\int_{T_k} \Big(f(z) - f(c) - (z - c) \cdot f'(c) \Big) \cdot dz = \int_{T_k} f(z) \cdot dz$$

and now the estimate (19) is this.

$$\left| \int_{T_k} f(z) \cdot dz \right| \leq \epsilon \cdot |T_k| \cdot |T_k|$$

We use (16) and (17) to replace T_k by T_0 in this last estimate.

$$\frac{1}{4^k} \cdot \left| \int_{T_0} f(z) \cdot dz \right| \leq \left| \int_{T_k} f(z) \cdot dz \right|$$

$$\leq \epsilon \cdot |T_k|^2 = \epsilon \cdot \left(\frac{1}{2^k} \cdot |T_0| \right)^2 = \epsilon \cdot \frac{1}{4^k} \cdot |T_0|^2$$

The estimate (18) follows, and we are done. □

10. Cauchy's Theorem

There are many versions of this fundamental theorem. We will employ the following kind of domain: a subset V of $\mathbb{C}$ is *star-like* if it is open and there is $b \in V$ such that for every $c \in V$, the line segment from b to c is entirely in V. In other words, the image of the smooth curve $L(b, c)$ lies in V. The point b is called a *base-point* for V; it does not have to be unique.

We will need the following technical fact.

Proposition 22. *Let V be a star-like subset of $\mathbb{C}$ with base point b. Let $c \in V$ and $r > 0$ and suppose that $D(c; r) \subseteq V$. Let $a \in D(c; r)$. Then the closed interior of the triangle $< L(b, a), L(a, c), L(c, b) >$ is contained in V.*

Proof. The line segments that are the images of $L(b, a)$ and $L(c, b)$ are in V since b is a base point. The line segment image of $L(a, c)$ lies in $D(c; r)$ and so it is in V, as well. Thus, if q is on one of the sides of the triangle, then the line segment from b to q is in V. If q is in the closed interior of the triangle, then it is on a line segment from b to a point on one of the sides; again, $q \in V$. □

Goursat's Theorem will combine with the following result to give Cauchy's Theorem.

Proposition 23. *Let V be a star-like open subset of $\mathbb{C}$ and let $f : V \to \mathbb{C}$ be continuous. Assume that $\int_T f(z) \cdot dz = 0$ for every chain triangle whose image is in V. Then $f(z)$ has an antiderivative in V.*

Proof. Let $b \in V$ be a base point. For each $z \in V$, we know that the image of $L(b, z)$ lies in V. Define

$$F(z) = \int_{L(b,z)} f(v) \cdot dv$$

so that $F : V \to \mathbb{C}$. We will prove that $F'(z) = f(z)$

Let $z \in V$ and choose $\epsilon > 0$. Because f is continuous at z, there is $\delta > 0$ such that if $|w - z| < \delta$, then we have $w \in D(b; r)$ and $|f(w) - f(z)| < \epsilon$.

Proposition 22 shows that the closed interior of the triangle

$$T =< L(b, z), L(z, w), L(w, b) >$$

is contained in the star-like set V. By hypothesis,

$$\int_T f(v) \cdot dv = 0$$

A previous problem showed that $L(w,b) = -L(b,w)$ and Property 3 gets involved as well. We also use the definition of F.

$$\begin{aligned}
0 &= \int_T f(v) \cdot dv \\
&= \int_{L(b,z)} f(v) \cdot dv + \int_{L(z,w)} f(v) \cdot dv + \int_{L(w,b)} f(v) \cdot dv \\
&= F(z) + \int_{L(z,w)} f(v) \cdot dv - \int_{L(b,w)} f(v) \cdot dv \\
&= F(z) - F(w) + \int_{L(z,w)} f(v) \cdot dv
\end{aligned}$$

and therefore

$$F(w) - F(z) = \int_{L(z,w)} f(v) \cdot dv \tag{20}$$

The number z is constant in the present context. Thus, we can compute

$$\int_{L(z,w)} f(z) \cdot dv = f(z) \cdot (w - z)$$

We can subtract each side of this equation from the opposite sides of (20).

$$F(w) - F(z) - f(z) \cdot (w - z) = \int_{L(z,w)} (f(v) - f(z)) \cdot dv \tag{21}$$

For v on the image of $L(z,w)$, we have $v \in D(z;\delta)$, and so we have $|f(v) - f(z)| < \epsilon$. Property 4 then takes (21) and gives this.

$$|F(w) - F(z) - f(z) \cdot (w - z)| \leq \epsilon \cdot |L(z,w)| = \epsilon \cdot |w - z|$$

That this inequality holds for all w in $D(z;\delta)$ and that ϵ is arbitrary show that $F'(z) = f(z)$, as needed. □

And now, one of the most remarkable theorems of mathematics.

Cauchy's Theorem. *Let V be a star-like subset of $\mathbb{C}$, and let f be holomorphic on V. Then f has an antiderivative on V. If G is a closed chain whose image is in V, then*

$$\int_G f(z) \cdot dz = 0$$

Proof. Goursat's Theorem tells us that $\int_T f(z) \cdot dz = 0$ for every chain triangle in V. Proposition 23 then finds an antiderivative $F(z)$ for $f(z)$. Then the statement about closed chains follows by Property 5. □

11. Null Chains and Equivalent Chains

Let V be an open subset of the complex numbers, and let G and H be chains whose images lie in V. We define $G + H$ to be their multi-set union: if $G =< g_1, \dots, g_j >$ and $H =< h_1, \dots, h_k >$, then

$$G + H =< g_1, \dots, g_j, h_1, \dots, h_k >$$

We could take some time to derive the obvious properties of this addition, we prefer merely to note that addition is obviously commutative and that

$$\int_{G+H} f(z) \cdot dz = \int_G f(z) \cdot dz + \int_H f(z) \cdot dz$$

for all continuous functions f on V.

A chain G is *null on* V if its image is contained in V and if $\int_G f(z) \cdot dz = 0$ for all f holomorphic on V. For instance, if $G =< g, -g >$ for a smooth curve g, then G is null by Property 3.

For a chain $G =< g_1, g_2, \dots, g_n >$, define $-G =< -g_1, \dots, -g_n >$, and we see that $G + (-G)$ is null.

Let W be a star-like subset of V. If G is a closed chain curve whose image is contained in W, then Cauchy's Theorem shows that G is null on V.

The sum of chains, each of which is null on V, is null on V.

If G, H are chains whose image lies in V, and if

$$\int_G f(z) \cdot dz = \int_H f(z) \cdot dz$$

for every holomorphic function f on V, then we say that G and H are *equivalent on* V.

A previous problem you did shows that if a, b, c are co-linear elements of $\mathbb{C}$, then $< L(a, b), L(b, c) >$ and $< L(a, c) >$ are equivalent on $\mathbb{C}$.

If there are chains A, B, each null on V, such that $G + A = H + B$, then it is easy to see that G and H are equivalent on V.

Here are two examples we will need; proofs and other examples will be discussed in class. As you read the statements of each of the next three problems, draw a picture of the hypothesis!

12. Cauchy's Integral Formula

This extremely powerful consequence of Cauchy's Theorem says that the values of a holomorphic function inside a circle are determined by the values on the circle.

Cauchy's Integral Formula. *Let V be an open subset of $\mathbb{C}$, and let f be holomorphic on V. Let $c \in V$ and suppose that $\overline{D}(c; r)$ is contained in V. If $z \in D(c; r)$, then*

$$f(z) = \frac{1}{2\pi i} \cdot \int_{C(c;r)} \frac{f(w)}{w - z} \cdot dw$$

Proof. Let $\epsilon > 0$. There is a positive number s such that $\overline{D}(z; s) \subseteq D(c; r)$ and such that $|f(w) - f(z)| \leq \epsilon$ for all $w \in \overline{D}(z; s)$.

A problem says that $C(c;r)$ and $C(z;s)$ are equivalent in $D(c;r) - \{z\}$. The function $f(w)/(w-z)$ is holomorphic there, and so

$$\int_{C(c;r)} \frac{f(w)}{w-z} \cdot dw = \int_{C(z;s)} \frac{f(w)}{w-z} \cdot dw \tag{22}$$

A problem you have done shows that

$$\int_{C(z;s)} \frac{dw}{w-z} = 2\pi i$$

and so since $f(z)$ is constant, we have

$$\int_{C(z;s)} \frac{f(z) \cdot dw}{w-z} = 2\pi i \cdot f(z)$$

Subtracting the two sides here from the opposite sides of (22), we obtain

$$\int_{C(c;r)} \frac{f(w)}{w-z} \cdot dw - 2\pi i \cdot f(z) = \int_{C(z;s)} \frac{f(w)-f(z)}{w-z} \cdot dw \tag{23}$$

We can estimate the integral on the right using the fact that $|f(w)-f(z)| \leq \epsilon$, from the choice of s, and that $|w-z| = s$.

$$\left| \int_{C(z;s)} \frac{f(w)-f(z)}{w-z} \cdot dw \right| \leq \frac{\epsilon}{s} \cdot 2 \cdot \pi \cdot s = \epsilon \cdot 2 \cdot \pi$$

Taking this to (23), we see that

$$\left| \int_{C(c;r)} \frac{f(w)}{w-z} \cdot dw - 2 \cdot \pi \cdot i \cdot f(z) \right| \leq \epsilon \cdot 2 \cdot \pi$$

Because ϵ is arbitrary, this gives the required formula. $\square$

Our next theorem involves this defintion: a set $V \subseteq \mathbb{C}$ is *path connected* if for every $a, b \in V$, there is a smooth curve starting at a, ending at b, and whose image lies in V.

The Maximum Modulus Theorem. *Let $f(z)$ be holomorphic on the path connected open set $V \subseteq \mathbb{C}$. Suppose that $|f(z)|$ has a maximum on V. Then $f(z)$ is constant on V.*

Proof. Suppose that the maximum of $|f(z)|$ occurs at $b \in V$. If this maximum is 0, then $f(z) = 0$ for all $z \in V$, and we are done. If $f(b) \neq 0$, then we replace $f(z)$ by $f(z)/f(b)$, so that $f(b) = 1$ and 1 is the maximum modulus.

Let M be the set of $z \in V$ such that $f(z) = 1$. We will show that $M = V$. The following claim does most of the work.

Claim 1. Let $c \in M$, and suppose that r is a positive number such that $\overline{D}(c;r) \subset V$. Then $f(z) = 1$ for all $z \in D(c;r)$.

Proof of Claim 1. By Cauchy's Integral Theorem, we have

$$1 = f(c) = \frac{1}{2\pi i} \cdot \int_{C(c;r)} \frac{f(w)}{w-c} \cdot dw$$

The curve is parametrized by $w = r \cdot \exp(i\theta) + c$, for $0 \le \theta \le 2\pi$. We see that

$$\frac{1}{2\pi i} \cdot \int_{C(c;r)} \frac{f(w)}{w-c} \cdot dw = \frac{1}{2\pi} \cdot \int_0^{2\pi} f(w) \cdot d\theta$$

Writing $f = u + i \cdot v$ in rectangular, and remembering $f(c) = 1$, we have

$$1 = \frac{1}{2\pi} \cdot \int_0^{2\pi} u(w) \cdot d\theta + \frac{i}{2\pi} \cdot \int_0^{2\pi} v(w) \cdot d\theta$$

Since u and v are real-valued, we conclude that

$$1 = \frac{1}{2\pi} \cdot \int_0^{2\pi} u(w) \cdot d\theta$$

We have $|u(w)| \le |f(w)| \le 1$ for each w. The function u is continuous on the circle, and so it must be identically 1 on the circle: $u(w) = 1$ for all $w = r \cdot \exp(i\theta) + c$. Since $|f(w)| \le 1$, we see that $f(w) = u(w)$ for all w, and so $v(w) = 0$ for all w. It follows that $f(w) = 1$ for all w.

If $z \in D(c;r)$, then Cauchy's Integral Formula tells us that

$$f(z) = \frac{1}{2\pi i} \cdot \int_{C(c;r)} \frac{f(w)}{w-z} \cdot dw = \frac{1}{2\pi i} \cdot \int_{C(c;r)} \frac{1}{w-z} \cdot dw = 1$$

■

To complete the proof, we show that $M = V$. If not, there is $c \in V \setminus M$. Because V is path connected, there is a smooth curve g starting at $b \in M$ and ending at c, with the image of g lying in V. Let $g : [p, q] \to V$, and then $g(p) = b$, so that $f(g(p)) = 1$, and $g(q) = c$, so that $f(g(q)) \neq 1$. Let s be the sup of $t \in [p, q]$ such that $f(g(t)) = 1$, and it is easy to see that $f(g(s)) = 1$ and that if $s < t \le q$, then $f(g(t)) \neq 1$.

There is a positive number r such that $D(g(s); r) \subset V$, and Claim 1 shows that $f(z) = 1$ on this disk. Since g is continuous, there is a real number t with $s < t \le q$ and $g(t) \in D(g(s); r)$. This is a contradiction, and it proves that $M = V$. □

We turn to a special case of what is called the *Residue Theorem*. This case allows us to compute some interesting integrals. The hypothesis of the following may be indicated by saying the $f(z)$ has a *simple pole* at w.

Proposition 24. *Let $w \in \mathbb{C}$ and $r > 0$ and suppose that $f(z)$ is holomorphic on $D(w;r) \setminus \{w\}$. Suppose that there is $A \in \mathbb{C}$ with $(z-w)f(z) \to A$ as $z \to w$. Let $0 < s < r$. Then*

$$\frac{1}{2\pi i} \cdot \int_{C(w;s)} f(z) \cdot dz = A$$

Proof. Let $\epsilon > 0$ and get $\delta > 0$ so that if $z \in \overline{D}(w;\delta) \setminus \{w\}$, then

$$|(z-w) \cdot f(z) - A| < \epsilon$$

We leave it to you to show that $C(w;s)$ is equivalent to $C(w;\delta)$ in $D(w;r) \setminus \{w\}$, and so

$$\frac{1}{2\pi i} \cdot \int_{C(w;s)} f(z) \cdot dz = \frac{1}{2\pi i} \cdot \int_{C(w;\delta)} f(z) \cdot dz$$

We know that

$$A = \frac{1}{2\pi i} \cdot \int_{C(w;\delta)} \frac{A}{z-w} \cdot dz$$

We know that the length of $C(w;\delta)$ is $2\pi\delta$, and for z on that circle, we can estimate

$$\left| \frac{(z-w) \cdot f(z) - A}{z-w} \right| \leq \frac{\epsilon}{\delta}$$

Thus, we can estimate

$$\begin{aligned} \left| \frac{1}{2\pi i} \cdot \int_{C(w;s)} f(z) \cdot dz - A \right| &= \left| \frac{1}{2\pi i} \cdot \int_{C(w;\delta)} f(z) \cdot dz - \frac{1}{2\pi i} \cdot \int_{C(w;\delta)} \frac{A}{z-w} \cdot dz \right| \\ &= \left| \frac{1}{2\pi i} \cdot \int_{C(w;\delta)} \frac{(z-w) \cdot f(z) - A}{z-w} \cdot dz \right| \\ &\leq \frac{1}{2\pi} \cdot 2\pi \cdot \delta \cdot \frac{\epsilon}{\delta} = \epsilon \end{aligned}$$

□

13. Holomorphic Functions are Analytic

Here is the promised converse to Theorem 19: holomorphic functions are analytic – they are represented by Taylor series on every disk in their domain! This is another central theorem of complex analysis.

Theorem 25. *Let f be holomorphic on the open set V in the complex plane, and let $\overline{D}(c;r) \subset V$. Then $f^{(k)}(c)$ exists for all $k \geq 0$, in fact*

$$f^{(k)}(c) = \frac{k!}{2\pi i} \cdot \int_{C(c;r)} \frac{f(z)}{(z-c)^{k+1}} \cdot dz$$

Furthermore,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} \cdot (z-c)^k$$

for every $z \in D(c;r)$.

Proof. The image of $C(c;r)$ is the circle of radius r about c. The function $f(z)$ is continuous on that circle, and so we can define

$$a_k = \frac{1}{2\pi i} \cdot \int_{C(c;r)} \frac{f(z)}{(z-c)^{k+1}} \cdot dz \quad \text{for each} \quad k \geq 0$$

Let $w \in D(c;r)$. Since $|w-c| < r$, there is a positive number $s < 1$ with $|w-c| < s \cdot r$. If z is on the image of $C(c;r)$, then $|z-c| = r$, and we can estimate

$$\left| \frac{w-c}{z-c} \right| \leq \frac{s \cdot r}{r} = s \tag{24}$$

Since $s < 1$, the geometric series

$$\sum_{k=0}^{\infty} \frac{(w-c)}{(z-c)^{k+1}}$$

converges. Moreover, the estimate (24) is independent of z, and so the convergence is uniform over the image of $C(c;r)$. Proposition 21 then allows us to permute the summation and integral sign in the following calculation.

$$\begin{aligned}
\sum_{k=0}^{\infty} a_k \cdot (w-c)^k &= \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \cdot \int_{C(c;r)} \frac{f(z)}{(z-c)^{k+1}} \cdot dz \right] \cdot (w-c)^k \\
&= \frac{1}{2\pi i} \cdot \int_{C(c;r)} \left[\sum_{k=0}^{\infty} \frac{(w-c)^k}{(z-c)^{k+1}} \right] \cdot f(z) \cdot dz \\
&= \frac{1}{2\pi i} \cdot \int_{C(c;r)} \left[\frac{1}{z-c} \cdot \frac{1}{1-(w-c)/(z-c)} \right] \cdot f(z) \cdot dz \\
&= \frac{1}{2\pi i} \cdot \int_{C(c;r)} \left[\frac{1}{z-c-(w-c)} \right] \cdot f(z) \cdot dz \\
&= \frac{1}{2\pi i} \cdot \int_{C(c;r)} \frac{1}{z-w} \cdot f(z) \cdot dz
\end{aligned}$$

By Cauchy's Integral Formula, this integral is $f(w)$, and we have the series formula we claimed.

The convergence of the series on $C(c;r)$ shows that the radius of convergence of a_k is at least r. Theorem 19 then shows that $f(z)$ is infinitely differentiable on $C(c;r)$, and that theorem also gives the claimed formula for the a_k. $\square$

The convergence of the series in the proof of Theorem 25 depends only on the fact that $f(z)/(z-c)^{k+1}$ is integrable over $C(c;r)$; we didn't use that $f(z)$ was holomorphic until we quoted Cauchy's Theorem. If we specify an integrable function on such a circle, the proof of the theorem shows how to define an analytic function on the disk from that function.

In Proposition 14, where we proved that the Cauchy-Riemann equations on real functions u, v lead to a holomorphic function $u + i \cdot v$, we assumed that the partial derivatives of u, v are continuous. As we will now see, Theorem 25 shows that the Cauchy-Riemann equations imply the continuity of the partial derivatives.

Theorem 26. *Let V be an open subset of $\mathbb{C}$, and let $f : V \to \mathbb{C}$. Then f is holomorphic on V if and only if its real and imaginary parts have continuous first partial derivatives that satisfy the Cauchy-Riemann equations.*

Proof. Write $f = u + i \cdot v$ with its real and imaginary parts.

Let $c \in V$ and let $r > 0$ with $D(c;r) \subset V$. Theorem 25 shows that there is a Taylor series for $f(z)$ in $D(c;r)$. Theorem 19 then says that $f'(z)$ is holomorphic and therefore continuous at c. We know that

$$f'(z) = \frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \cdot \frac{\partial u}{\partial y}$$

Since f' is continuous, we see that the partial derivatives just written are continuous at c. Thus, if f is holomorphic, then its real and imaginary parts have continuous first partial derivatives.

Proposition 14 proves that if u, v have continuous first partial derivatives, and if they satisfy the Cauchy-Riemann equations, then f is holomorphic. $\square$

But more is true. Since $f(z)$ has a Taylor series, it has infinitely many derivatives, all of which are holomorphic. It follows that u and v have continuous partial derivatives of all orders. If you are at all familiar with partial differential equations, you will find it very remarkable that the assumption of a first-order equation leads to infinite-order continuity.

Morera's Theorem is a kind of converse to Cauchy's Theorem. We are working with star-like subsets of the complex numbers; there are more general versions of this theorem.

Morera's Theorem. *Let V be a star-like subset of $\mathbb{C}$, and let $f : V \to \mathbb{C}$. Suppose that $\int_G f(z) \cdot dz = 0$ for every closed chain G in V. Then f is holomorphic in V.*

Proof. Proposition 23 constructs an antiderivative $F(z)$ for $f(z)$. By Theorem 25, the function $F(z)$ is analytic, and by Theorem 19, $F''(z)$ exists. This shows that $f'(z)$ exists, and so $f(z)$ is holomorphic. $\square$

Recall that an entire function is a holomorphic function over the entire complex plane. Here is another famous consequence of Theorem 25.

Liouville's Theorem. *Every bounded entire function is constant.*

Proof. Let $f(z)$ be an entire function and suppose there is a real number B such that $|f(z)| \leq B$ for all $z \in \mathbb{C}$. Let $w \in \mathbb{C}$ and choose a positive real number R. Theorem 25 shows that

$$f'(w) = \frac{1}{2\pi i} \cdot \int_{C(w;R)} \frac{f(z)}{(z-w)^2} \cdot dz$$

Since the length of $C(w; R)$ is $2\pi R$, we can estimate this integral

$$\left| \frac{1}{2\pi i} \cdot \int_{C(w;R)} \frac{f(z)}{(z-w)^2} \cdot dz \right| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot B \cdot \frac{1}{R^2} = \frac{B}{R}$$

This quantity goes to 0 as $R \to \infty$, and we conclude that $f'(w) = 0$. Proposition 13 shows that f is constant. $\square$

Here is a remarkable uniqueness theorem: distinct holomorphic functions cannot agree too often.

Theorem 27. *Let $f(z)$ and $g(z)$ be holomorphic on the path connected open set $V \subseteq \mathbb{C}$. Suppose that z_k for $k \geq 0$ is a sequence of distinct elements of V that converges to an element of V, and suppose that $f(z_k) = g(z_k)$ for all $k \geq 0$. Then $f(z) = g(z)$ for every $z \in V$.*

Proof. We can define $h(z) = f(z) - g(z)$ and prove that $h(z)$ is identically 0, given that $h(z_k) = 0$ for all k.

Let w be the limit of the z_k, and since h is continuous, we have $h(w) = 0$. Let M be the set of $a \in V$ such that there is a sequence of distinct elements $a_k \in V$ converging to a and for which $h(a_k) = 0$ for all k. We see that $w \in M$.

Claim. If $a \in M$, and if $r > 0$ with $D(a;r) \subset V$, then $h(z) = 0$ for every $z \in D(a;r)$. Also, $D(a;r) \subseteq M$.

Proof of the Claim. Let a and r be as hypothesized. By Theorem 25 we can write $h(z)$ as a series in $D(a;r)$.

$$h(z) = \sum_{k=0}^{\infty} b_k \cdot (z-a)^k$$

Since $h(a) = 0$, we have $b_0 = 0$. Suppose that h is not identically 0 on $D(a;r)$, and then some b_k is not 0. Let j be the minimal such k, and we can write

$$h(z) = b_j \cdot (z-a)^j + (z-a)^{j+1} \cdot g(z)$$

where $g(z)$ is holomorphic on $D(a;r)$.

The function g is continuous on the closed and bounded set $\overline{D}(a;r/2)$. Proposition 10 then finds a positive real number $B \geq |g(z)|$ on that set. Choose a positive number $s \leq r/2$ such that $s \cdot B < |b_j|/2$.

Let $z \in D(a;s)$. We can estimate

$$\begin{aligned}
|h(z)| &= \left|b_j \cdot (z-a)^j + (z-a)^{j+1} \cdot g(z)\right| \\
&= |z-a|^j \cdot |b_j + (z-a) \cdot g(z)| \\
&\geq |z-a|^j \cdot [|b_j| - |z-a| \cdot |g(z)|] \\
&\geq |z-a|^j \cdot [|b_j| - s \cdot B] \\
&\geq |z-a|^j \cdot [|b_j| - |b_j|/2] \\
&\geq |z-a|^j \cdot |b_j|/2
\end{aligned}$$

We see that if $z \neq a$, then $h(z)$ cannot be 0. In other words, a is the only element of $D(a;s)$ where h is 0. But this contradicts the existence of a sequence of *distinct* elements of V converging to a on which h is 0.

Every element of $D(a;r)$ is a limit of a sequence of distinct elements of $D(a;r)$. The function h is 0 on this sequence, and so $D(a;r) \subseteq M$. This proves the Claim. ■

Now we can show that $M = V$, so that $h(z) = 0$ for all $z \in V$. Let $a \in M$, as before, and let $b \in V$. There is a smooth curve $g : [0,1] \to V$ such that $g(0) = a$ and $g(1) = b$. Let T be the set of all $t \in [0,1]$ such that $g(t) \in M$. Then $g(0) = a \in M$, so that $0 \in T$. It follows that T has a sup s.

There is $r > 0$ such that $D(g(s);2r) \subseteq V$. Get $\delta > 0$ such that if $|t-s| < \delta$, then $g(t) \in D(g(s);r)$. Becuse s is the sup of T, there is $t \in (s-\delta, s] \cap T$. We claim that $g(s) \in D(g(t);r) \subseteq V$. Indeed, since $g(t) \in D(g(t);r)$, we have $g(s) \in D(g(t);r)$. If $z \in D(g(t);r)$, then

$$|z - g(s)| \leq |z - g(t)| + |g(t) - g(s)| < r + r = 2 \cdot r$$

In other words, $D(g(t);r) \subseteq D(g(s);2r) \subseteq V$, as claimed.

Now we apply the earlier claim to $D(g(t);r)$, using that $g(t) \in M$. We see that $D(g(t);r) \subseteq M$. In particular $g(s) \in M$. If $s < 1$, then there is s_1 with $s < s_1 < s+\delta$, and then $g(s_1) \in D(g(s);r) \subseteq M$. This contradicts the fact that s is the sup of T. Thus, $s = 1$, and so $b = g(1) \in M$. □