math 535, Spring 2016 – Parks

COMPLEX ANALYSIS PROBLEMS

The sections correspond to the sections of the Notes.

2. The Complex Plane

1. Find a polar representation for the following complex numbers: 1, -1, i, $-i$, $-3+3\cdot i$, $\sqrt{3}-i$.

2. Prove that $\exp(i\cdot\alpha)^n = \exp(i\cdot n\cdot\alpha)$ for all $\alpha\in\mathbb{R}$ and $n\in\mathbb{Z}$. (Hint: induction on n for $n\geq 0$. Then do the case $n<0$.)

3. Use DeMoivre's Theorem to find the rectangular representation of the three numbers w such that $w^3 = i$. Your representation should not involve the cosine or sine functions.

4. Find the rectangular representation for the four numbers w such that $w^4 = -16$. Your representation should not involve the cosine or sine functions.

5. Show that $\overline{1/z} = 1/\overline{z}$ for all non-zero complex numbers z.

6. Let $M>0$. Define the *open square* $S(M)$ to be the set of $x+i\cdot y$ with $x,y\in\mathbb{R}$ such that $|x|<M$ and $|y|<M$. Let $A\subseteq\mathbb{C}$. Show that A is bounded if and only if there is $M>0$ such that if $A\subseteq S(M)$.

7. Let $z\in\mathbb{C}$. Show that $\mathbb{C}\setminus\{z\}$ is open.

8. Prove that $\overline{D}(z;r)$ is closed in $\mathbb{C}$ for all $z\in\mathbb{C}$ and $r>0$.

9. Prove that every finite subset of the complex numbers is closed. (Hint: easier if you use Proposition 7.)

3. Limits and Continuity

10. Let $f:\mathbb{R}\to\mathbb{C}$, and let $c\in\mathbb{C}$. Apply the ϵ,δ definition without the requirement of the limit point to show that

$$\lim_{z\to i} f(z) = c$$

(In other words, since i is not a limit point of $\mathbb{R}$, the "limit" is arbitrary.)

11. Let $A\subseteq\mathbb{C}$ and $f:A\to\mathbb{C}$ and let c be a limit point of A. Write $f=u+i\cdot v$, its real and imaginary parts, so that $u,v:A\to\mathbb{R}$. Write $L=L_1+i\cdot L_2$, where $L_1,L_2\in\mathbb{R}$. Show that

$$\lim_{z\to c} f(z) = L \quad \text{if and only if} \quad \lim_{z\to c} u(z) = L_1 \quad \text{and} \quad \lim_{z\to c} v(z) = L_2$$

12. Let $f(z)$ and $g(z)$ be non-zero polynomials of the same degree n, and let f_n and g_n be their respective leading coefficients. Show that

$$\lim_{|z|\to\infty} \frac{f(z)}{g(z)} = \frac{f_n}{g_n}$$

(Hint: use the $1/z$ limit.)

13. Let $A \subseteq \mathbb{C}$ and let $f : A \to \mathbb{C}$ be continuous. Let $w \in \mathbb{C}$ and let T be the set of $z \in A$ such that $f(z) = w$. Show that T is a closed subset of $\mathbb{C}$.

4. The Derivative

14. Let n be an integer and define $f(z) = z^n$ for all $z \in \mathbb{C}$ (have $z \neq 0$ when $n < 0$). Show that $f'(z) = n \cdot z^{n-1}$ by following these steps.

(a) Calculate the derivative of $z^0 = 1$ from the definition.
(b) Use induction and product rule to show get the formula for $n \geq 0$.
(c) Use the quotient rule to get the formula for $n < 0$.

5. The Cauchy-Riemann Equations

15. Verify the Cauchy-Riemann equations for $f(z) = z^3 - 2 \cdot z$.

16. Verify the Cauchy-Riemann equations for $f(z) = 1/z$. (Hint: we have formulas for the real and imaginary parts of $1/z$.)

17. Let $w \in \mathbb{C}$ and $r > 0$, and let $f : D(w; r) \to \mathbb{R}$ be holomorphic. Show that f is constant. (Hint: Cauchy-Riemann and equation (1) on p.2 of the Notes.

18. Let $f(x + i \cdot y) = x \cdot y$, for all $x, y \in \mathbb{R}$. Define A to be the set of $x + i \cdot x$, for all $x \in \mathbb{R}$. Show that if we consider $f : A \to \mathbb{R}$, then $f'(x + i \cdot y) = 2 \cdot x$. (Note: it follows from the previous problem that if we consider $f : \mathbb{C} \to \mathbb{R}$, then it does not have a derivative; in other words, f is not holomorphic.)

19. Show that $u(x, y) = e^x \cdot \cos(y)$ and $v = e^x \cdot \sin(y)$ satisfy the conditions of Proposition 14 on $\mathbb{C}$. (The holomorphic function that results will be seen to be the exponential function $\exp(z) = e^z$.)

20. Let V be the (open) set of all complex numbers with positive real part. For $x + i \cdot y \in V$, define $v(x, y) = \arctan(y/x)$. Define $u(x, y) = \ln(\sqrt{x^2 + y^2})$. Show that u, v satisfy the conditions of Proposition 14 on V. (The holomorphic function that results is the natural logarithm – the inverse function to the exponential.)

6. Taylor Series: Analytic Functions

21. If a_n is non-zero constant, then every positive number $r \leq 1$ is a radius for a_n. If $a_n = t^n$ for some non-zero complex number t, then r is a radius if and only if $r \leq 1/|t|$.

22. Show that the given sequences have the claimed ratio limit as in the Ratio Test.

sequence	limit	notes
1	1	
n^k	1	k is constant
$1/n!$	∞	
r^n	$1/\lvert r\rvert$	r is a non-zero constant

7. Exponential, Cosine, Sine, Logarithm

23. Show that $\sin(z) = 0$ if and only if $z = \pi \cdot k$ for some integer k.

24. Define

$$\cosh(z) = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh(z) = \frac{e^z - e^{-z}}{2}$$

(Note: cosh is pronounced *coash*, and sinh is *sinch*.)

(a) Show that $\cosh(0) = 1$ and $\sinh(0) = 0$ and that $\cosh^2(z) - \sinh^2(z) = 1$.
(b) Show that $\cosh'(z) = \sinh(z)$ and $\sinh'(z) = \cosh(z)$.
(c) Find (nice) series formulas for $\cosh(z)$ and $\sinh(z)$.
(d) Show that $\cos(i \cdot z) = \cosh(z)$ and $\sin(i \cdot z) = i \cdot \sinh(z)$.

25. Use the previous problem to find the real and imaginary parts of $\cos(x + i \cdot y)$ and $\sin(x + i \cdot y)$, where $x, y \in \mathbb{R}$.

26. Show that $L(x+iy) = \ln|(x,y)| + i \cdot \arctan(y/x)$ maps S_1 onto T_1, where T_1 is the set of complex numbers $a + i \cdot b$ such that $-\pi/2 < b < \pi/2$.

27. Let $A(x,y) = \cot^{-1}(x/y)$ for $(x,y) \in S_2$. Show that

$$\frac{\partial A}{\partial x} = -\frac{y}{x^2+y^2} \quad \text{and} \quad \frac{\partial A}{\partial y} = \frac{x}{x^2+y^2}$$

Thus, $u(x,y) = \ln\sqrt{x^2+y^2}$ along with $A(x,y)$ satisfy the Cauchy-Riemann equations and have continuous first partial derivatives on S_2.

28. Show that $\log'(z) = 1/z$. (Hint: $\log(\exp(z)) = z$).)

29. Show that the function log on S_1 agrees with log on S_2, on $S_1 \cap S_2$. (The set $S_1 \cap S_2$ is the open first quadrant!)

30. Let $w \in S$, and let $w = x + i \cdot y$ in rectangular. We want to find $r > 0$ with $D(w; r) \subset S$. Show that if $x \geq 0$, we can let $r = |w|$, whereas if $x < 0$, we can use $r = |y|$.

31. Let $w \in S$ and get $r > 0$ with $D(w; r) \subset S$. Show that if $z \in D(w; r)$, then $z/w \in D(1; 1)$.

32. Let $w \in S$ and get $r > 0$ with $D(w; r) \subset S$. Show that if $z \in D(w; r)$, then $\log(z/w) = \log(z) - \log(w)$, using the following steps.

(a) Since $z/w \in D(1; 1)$, the number $\log(z/w)$ is defined. Use the Chain Rule to show that $\log(z/w)$ has the same derivative as $\log(z)$. (Here, z is the variable and w is constant.)
(b) Conclude (why?) that $\log(z) + C = \log(z/w)$ for some constant C.
(c) Use $z = w$ to determine C.

33. Observe that $\exp(3\pi i/4) \in S$, and that its square is in S. Show that

$$\log\left([\exp(3\pi i/4)]^2\right) \neq 2 \cdot \log(\exp(3\pi i/4)$$

(Note: in the real numbers $\ln(x^2) = 2 \cdot \ln(x)$ for all $x > 0$.)

34. For the function $R(z)$ defined on p.21 of the Notes, show that the real and imaginary parts of $R(z) = u + i \cdot v$ are these:

$$u(x, y) = \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} \quad \text{and} \quad v(x, y) = \frac{y}{2 \cdot u}$$

(Hint: start by showing that $u > 0$ for all $z \in S$. Find v first.)

35. Define $L(z) = \log(z) + 2\pi i$, for all $z \in S$. Show that $L(z)$ is also a logarithm: $L(\exp(z)) = z$ and $\exp(L(z)) = z$. Use $L(z)$ instead of $\log(z)$ to define the square root $R_1(z)$. What is $R_1(z)$ in terms of the square root $R(z)$ defined using $\log(z)$?

36. If α is a non-negative integer, then $\binom{\alpha}{n} = 0$ for all $n > \alpha$, and every positive number is a radius for the sequence. Otherwise, all the terms are non-zero and 1 is a radius. (Hint for the case that α is not a non-negative integer: Ratio Test.)

37. For $\alpha \in \mathbb{C}$ and $w \in S$. Get $r > 0$ with $D(w; r) \subset S$, as in previous problems. Let $z \in D(w; r)$. Show that $(z/w)^\alpha = (z^\alpha)/(w^\alpha)$. (Hint: a previous problem said something about $\log(z/w)$.)

38. Carefully complete the argument that formula (10) on p.22 of the Notes is correct.

39. Prove the following formula of Newton.

$$z^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2n)!}{4^n \cdot n! \cdot n!} \cdot (z-1)^n \quad \text{for all} \quad z \in D(1;1)$$

40. Do we have the following identity?

$$(z^\alpha)^\beta = z^{\alpha \cdot \beta}$$

(where $z \in S$ and $\alpha, \beta \in \mathbb{C}$) Prove the identity, or find a counterexample.

8. Line Integrals

41. Let $g : [a,b] \to \mathbb{C}$ be a smooth curve. Define $h(t) = g(a + t \cdot (b-a))$ for $t \in [0,1]$, and show that h is a smooth curve with the same image as g. (Thus, we can assume that the domain of a smooth curve is $[0,1]$, if we find that convenient.)

42. Let $g : [a,b] \to \mathbb{C}$ be a smooth curve. Prove that $|g| = |-g|$. (Note: remember that $-g$ is not multiplication by -1, but *running g backwards*.)

43. Show that the length of $L(p,q)$ is $|q-p|$.

44. Show that $-L(p,q) = L(q,p)$.

45. Prove that $|C(p;r)| = 2 \cdot \pi \cdot r$.

46. Let $c \in \mathbb{C}$ and let r be a positive real number. Show that

$$\int_{C(c;r)} \frac{dz}{z-c} = 2\pi i$$

47. Evaluate these line integrals.

a) $\displaystyle\int_{C(0;r)} z \cdot dz$ **b)** $\displaystyle\int_{C(0;r)} \overline{z} \cdot dz$

48. Let a, b, c be co-linear elements of $\mathbb{C}$. Let $f(z)$ be continuous on the line through these points. Show that

$$\int_{L(a,b)} f(z) \cdot dz + \int_{L(b,c)} f(z) \cdot dz = \int_{L(a,c)} f(z) \cdot dz$$

(Hint: there is $q \in \mathbb{R}$ such that $b = a + q \cdot (c-a)$. Then $b = c - (1-q) \cdot (c-a)$. Substitute $s = q \cdot t$ in the integrals over $L(a,b)$ and in $L(b,c)$.)

49. For $0 \le t \le 1$, define $g(t) = t + i \cdot t$ and $h(t) = t + i \cdot t^2$, so that both g and h start at 0 and end at $1 + i$. Define $P(x + i \cdot y) = x \cdot y$ and $Q(z) = z^2$. Evaluate these line integrals.

(a) $\displaystyle\int_g P \cdot dz$ **(b)** $\displaystyle\int_h P \cdot dz$

(c) $\displaystyle\int_g Q \cdot dz$ **(d)** $\displaystyle\int_h Q \cdot dz$

50. Let $G =< L(-1, -i), L(-i, 1), \mathrm{arc}(0, 1, 0, \pi) >$. Compute

a) $\displaystyle\int_G e^z \cdot dz$ **b)** $\displaystyle\int_G \frac{dz}{z}$

(Hint: Property 5 can be applied to the integral over each smooth curve piece. This makes (a) easy! In (b), be careful with the branch of the logarithm – you can't use the same one on each piece.)

9. Goursat's Theorem

51. Prove a version of Goursat's Theorem where a chain rectangle is used in place of a chain triangle.

10. Cauchy's Theorem

52. Let $c \in \mathbb{C}$ and let r be a positive real number. Show that $D(c; r)$ is star-like.

53. Let V be a star-like subset of $\mathbb{C}$, and suppose that a, b are base points. Show that every point on the line segment from a to b is a base point.

54. In Section 7 of the Notes we defined the set S consisting of those complex numbers that are not non-positive real numbers. Show that S is star-like.

55. Let V be the set of $x + i \cdot y \in \mathbb{C}$ such that either $x \neq 0$ or $y > 0$. Show that the set V is star-like with base point i, and that $1/z$ is holomorphic on V. We will find a nice formula for the function

$$F(z) = \int_{L(i,z)} \frac{dv}{v} \quad \text{for all} \quad z \in V$$

(Of course, $F(z)$ will be a logarithm.) Let $z \in V$, let $\theta \in \mathrm{Arg}(z)$ with $-\pi/2 < \theta < 3\pi/2$, and show that the following is a closed chain in V:

$$< L(i, z),\ \mathrm{arc}(|z|, \alpha, 0),\ L(|z|, 1),\ \mathrm{arc}(1, 0, \pi/2) >$$

Use this closed chain along with Cauchy's Theorem to compute the integral over $L(i, z)$ in terms of integrals that are easy to compute.

11. Null Chains and Equivalent Chains

56. Let $c \in \mathbb{C}$ and let $p > 0$. Let $0 < r < q < p$. Then $C(c;r)$ is equivalent to $C(c;q)$ on $D(c;p) \setminus \{c\}$.

57. Let $c \in \mathbb{C}$ and let $p > 0$, and let V be an open set containing $\overline{D}(c;p)$. Let $b \in \overline{D}(c;p)$. Let $r > 0$ with $C(b;r)$ contained in $\overline{D}(c;p)$. Then $C(c;p)$ and $C(b;r)$ are equivalent on $V \setminus \{b\}$.

58. Show that $< C(0;3) >$ is equivalent to $< C(-1;1), C(1;1) >$ on $\mathbb{C} \setminus \{-1, 1\}$.

59. Evaluate the following without computing any antiderivatives.

$$\int_{C(0;3/2)} \frac{dz}{z^2 + z - 2}$$

12. Cauchy's Integral Formula

60. Let

$$G =< L(1,i), L(i,-1), L(-1,-i), L(-i,1) >$$

The image of G is a square. Compute

$$\int_G \frac{dz}{z^3}$$

(Hint: show that G is equivalent to $C(0;1)$ on the domain of $1/z^3$.)

61. Evaluate the following.

$$\int_{C(0;3)} \frac{z}{z^2 - 1} \cdot dz$$

(Hint: A previous problem showed that $C(0;3)$ is equivalent to two circles centered at $-1, 1$, respectively.)

62. Evaluate the following integrals. We will use the triangle Q:

$$Q =< L(\text{-}1 + 3i/2, 3/2\text{-}i), L(3/2\text{-}i, 3/2 + 3i/2), L(3/2 + 3i/2.\text{-}1 + 3i/2) >$$

(a) $\displaystyle\int_{C(\pi;1)} \frac{1}{\sin(z)} \cdot dz$ **(b)** $\displaystyle\int_Q \frac{1}{z^2+1} \cdot dz$

(c) $\displaystyle\int_{C(0;1)} \frac{1}{\exp(z) - 1} \cdot dz$ **(d)** $\displaystyle\int_Q \frac{z}{\log(z)} \cdot dz$

(Note: don't forget to prove that the relevant function is holomorphic where you need it to be.) (Hint for (a), (c), and (d): the appropriate limits can be calculated from series without using L'Hospital's Rule – which latter fact hasn't been proved.)

63. Evaluate the following integral. (Hint: separate the two singularities.)

$$\int_{C(\pi/2;2)} \frac{\exp(z)}{\sin(z)} \cdot dz$$

64. A star-like set is path connected. There are path connected open sets that are not star-like.

13. Holomorphic Functions are Analytic

65. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = -x^2$ when $x < 0$, and $f(x) = x^2$ when $x \geq 0$. Show that $f(x)$ is differentiable but that $f''(0)$ does not exist. (This simple example shows that nothing like Theorem 25 can be true in the reals.)

66. Let $g(z)$ be defined on the unit circle: $g(z) = 0$ for z below the x-axis; $g(z) = 1$ on or above the x-axis. Define

$$a_k = \frac{1}{2\pi i} \cdot \int_{C(0;1)} \frac{g(z)}{z^{k+1}} \cdot dz$$

Find a nice formula for the a_k to make it obvious that $\sum_{k=0}^{\infty} a_k \cdot z^k$ is holomorphic on $D(0;1)$.

67. Show that $\sin(1/z)$ and the constant function 0 agree on a converging sequence of z-values. Yet the two functions are not equal. Why does that not violate Theorem 27?